THE CONJUGATE FOURIER-STIELTJES INTEGRAL IN THE PLANE[1]

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Let $K(x)$ with $x=(x_1, x_2)$ be a Lip $(\alpha, 2)$ conjugate Calderon-Zygmund kernel with $1/2<\alpha<1$, i.e. $K(x)=\Omega(\theta)r^{-2}$ where (r, θ) are the usual polar coordinates of x with $\Omega(\theta)$ a continuous periodic function of period 2π with vanishing integral over the interval $[0, 2\pi]$ satisfying the condition $\int_0^{2\pi}[\Omega(\theta+h)-\Omega(\theta)]^2 d\theta=0(h^{2\alpha})$ as $h\rightarrow 0$ (See [2] and [7, p. 106].) Let F be a countably additive set function defined on the Borel sets of the plane having finite total variation. Furthermore let $f(y)=(2\pi)^{-2}\int_{E_2}e^{-i(y,x)}dF(x)$ be the Fourier-Stieltjes transform of F with E_2 the plane and (y, x) the usual scalar product. Also let $k(y)$ be the principal-valued Fourier transform of K, i.e. $k(y)=(2\pi)^{-2}\lim_{t\rightarrow 0;\lambda\rightarrow\infty}\int_{D(0,\lambda)-D(0,t)}e^{-i(y,x)}K(x)dx$ where $D(x, t)$ represents the open disc with center x and radius t. (It follows from the above assumptions that $k(y)$ exists for every y.) Then formally the conjugate Fourier-Stieltjes integral of F is given by $4\pi^2\int_{E_2}e^{i(y,x)}f(y)k(y)dy$. In [2, p. 118], it is shown that $\lim_{t\rightarrow 0}\int_{E_2-D_2(x,t)}K(x-y)dF(y)$ exists and is finite almost everywhere. We call this limit the conjugate of F with respect to K and designate it by $\tilde{F}(x)$. With $|y|=(y_1^2+y_2^2)^{1/2}$ and $I_R(x)=4\pi^2\int_{E_2}e^{-|y|/R}e^{i(y,x)}f(y)k(y)dy$, we propose to prove in this note the following theorem:

THEOREM 1. $\lim_{R\rightarrow\infty} I_R(x)=\tilde{F}(x)$ *almost everywhere.*

In a certain sense this result is the planar analogue of [7, p. 54]. In a forthcoming paper we shall extend this result to n-dimensional Euclidean space and the n-dimensional torus. We shall also study those kernels which are Bochner-Riesz summable almost everywhere. In particular we shall show that if $K(x)$ is in C^∞ then the conjugate Fourier-Stieltjes integral of F is summable (R, α) for $\alpha>1/2$ to $\tilde{F}(x)$ almost everywhere, thus paralleling Bochner's result [1] for the Fourier-Stieltjes integral of F.

Letting $D_{sym}F$ designate the symmetric derivative of F [5, p.149] and $\int_B|dF|$ the total variation of F over B, we observe from [5, p. 119 and p. 152] and the standard argument of Lebesgue that

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