CONTINUATION AND REFLECTION OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS[1]

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A solution of an ordinary differential equation can be continued as long as its graph stays in the domain, in which the equation is regular. On the other hand a solution of a partial differential equation can have a *natural boundary* interior to the domain of regularity of the equation. Let R be a closed region and S a portion of the boundary of R. Then S is a natural boundary for a solution u defined in R, if there exists no solution defined in a full neighbourhood of a point of S which agrees with u in R. Examples for the occurrence of such natural boundaries are well known from the theory of harmonic functions. Neither the equation nor the solution has to show any very singular behavior on approaching a natural boundary S.

Only in very special situations can one prove that *every* sufficiently regular solution known in a region R can be continued across a portion S of the boundary. This is e.g. the case for solutions of a single differential equation which is hyperbolic with respect to S. It is also the case for solutions of certain *overdetermined* systems of equations, like those associated with analytic functions of several complex variables; other systems with this property have been studied by S. Bochner [1].

In more general cases the only solutions for which one can prove continuibility are those satisfying *suitable boundary* conditions on S. The classical example is furnished by solutions $u(x, y)$ of the Laplace equation

$$u_{xx} + u_{yy} = 0 \tag{1a}$$

defined for $y \geqq 0$ and satisfying the boundary condition

$$u(x, 0) = 0. \tag{1b}$$

They can always be continued across $y = 0$ by the formula

$$u(x, -y) = - u(x, y). \tag{1c}$$

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[1] A summary of the results of this paper appeared in *Colloques internationaux du centre national de la recherche scientifique*, vol. 71, Nancy, 1956.