ON THE CHARACTERS OF A SEMISIMPLE LIE GROUP

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Let G be a connected semisimple Lie group and let Z denote its center. If π is a representation [2c] of G on a Hilbert space $\mathfrak{H}$ we consider the space V consisting of all finite linear combinations of elements of the form

$$\int f(x)\pi(x)\psi dx \qquad (f \in C_c^\infty(G),\ \psi \in \mathfrak{H}),$$

where dx is the Haar measure of G and $C_c^\infty(G)$ is the set of all (complex-valued) functions on G which are everywhere indefinitely differentiable and which vanish outside a compact set. V is called the Gårding subspace of $\mathfrak{H}$. Let R and C be the fields of real and complex numbers respectively and $\mathfrak{g}_0$ the Lie algebra of G. We complexify $\mathfrak{g}_0$ to $\mathfrak{g}$ and denote by $\mathfrak{B}$ the universal enveloping algebra of $\mathfrak{g}$ [2a]. Then there exists a (uniquely determined) representation π_V of $\mathfrak{B}$ on V such that $\pi_V(X)\psi = \lim_{t\to 0} (1/t)\{\pi(\exp tX)\psi - \psi\}$ $(X \in \mathfrak{g}_0,\ \psi \in V,\ t \in R)$. Let $\mathfrak{Z}$ denote the center of $\mathfrak{B}$. We say that π is quasi-simple if there exist homomorphisms η and χ of Z and $\mathfrak{Z}$ respectively into C such that $\pi(\zeta)\phi = \eta(\zeta)\phi$, $\pi_V(z)\psi = \chi(z)\psi$ for all $\zeta \in Z$, $z \in \mathfrak{Z}$, $\phi \in \mathfrak{H}$ and $\psi \in V$. η is then called the central character and χ the infinitesimal character of π. An irreducible unitary representation is automatically quasi-simple [5].

Let A be a bounded linear operator on $\mathfrak{H}$. We say that A is of the trace class or A has a trace if for every complete orthonormal set $(\psi_j)_{j\in J}$ in $\mathfrak{H}$ the series[1] $\sum_{j\in J} (\psi_j, A\psi_j)$ converges absolutely and its sum is independent of the choice of the complete orthonormal set.[2] We call this sum the trace of A and denote it by $\operatorname{Sp} A$. Now suppose π is quasi-simple and irreducible. Then it can be shown (see [2e]) that for any $f \in C_c^\infty(G)$ the operator $\int f(x)\pi(x)dx$ is of the trace class. If we denote its trace by $T_\pi(f)$ we get a linear function T_π on $C_c^\infty(G)$ which is actually a distribution (see [4; and 2e]). We call this distribution the character of π. Our object is to try to determine T_π.

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[1] As usual (ϕ, ψ) denotes the scalar product of the two elements ϕ and ψ in $\mathfrak{H}$.

[2] Actually it can be shown that this independence of the sum follows automatically from the absolute convergence of the series for every orthonormal base.