HIERARCHIES OF NUMBER-THEORETIC PREDICATES

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The existence of hierarchies of point sets in analysis has long been familiar from the work of Borel and Lusin. The study of the hierarchies in number theory which we consider here began with a theorem presented to the Society in 1940 and published in [12]. These hierarchies have applications in foundational investigations, but we shall be concerned here with the exploration of their structure (using classical logic). We shall survey the previous results from the beginning, and conclude with a few new ones. We have endeavored to make the exposition complete enough so that the layman in this field can get the gist of the arguments without consulting the references.

1. Recursive functions and predicates.

By a *number-theoretic function* (*predicate*) we mean a function, of independent variables ranging over the natural numbers $0, 1, 2, \cdots, x, x+1, \cdots$, taking natural numbers (propositions, true or false) as values.

By *general recursive* functions (predicates) we mean ones whose values can be computed (decided) by ideal computing machines not limited in their space for storing information. A theory of such machines was given by Turing [30] and in less detail by Post [24] (also cf. [16, Chapter XIII]). The general recursive functions can also be described as those whose values can be expressed by equations derivable formally from "recursion equations" defining the functions, in a sense first formulated precisely by Gödel [9] who built on a suggestion of Herbrand's (also cf. Church's [3], and our [10; 12; 16, Chapter XI]). A *general recursive* predicate $P(x_1, \cdots, x_n)$ is then one whose *representing function* $\phi(x_1, \cdots, x_n)$ ($=0$ or 1 according as $P(x_1, \cdots, x_n)$ is true or false) is general recursive.

The computation of a value of a general recursive function may involve a search through the natural numbers for the least one y with a given property without a bound for such a y having already been computed. By allowing such searches also when they may not terminate, we obtain an extension of the class of the general recursive functions to the *partial recursive* functions, which need not be defined for all sets of arguments [11; 12; 16, Chapter XII]. By disallowing such

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