GRADIENT MAPPINGS

E. H. ROTHE

1. **Introduction.** A gradient field in a finite-dimensional Euclidean space is a field of vectors $g(x)$ for which there exists a scalar function $I(x)$ such that $g(x) = \text{grad } I(x)$. A classical example is a conservative force field in which case $I(x)$ is the potential. It has been known for a long time that the treatment of such fields offers a considerable number of simplifications and special properties not shared by arbitrary vector fields. On the other hand those boundary value and integral equation problems which can be derived from variational problems likewise offer simplifications and special properties not shared by more general problems of this type. Now the relation between an integral in the calculus of variations which is to be made an extremum and the corresponding Euler-Lagrange equation, or rather the operator given by the Euler-Lagrange expression, is quite analogous to the relation between the scalar $I(x)$ and its gradient. It seems therefore reasonable to expect that with a proper definition of the term gradient one will be able to obtain a theory which encompasses the finite-dimensional as well as the function space case.[1]

2. **The definition of the term gradient mapping.** If $I(x)$ is a differentiable scalar defined for points $x = (x_1, x_2, \cdots, x_n)$ of a Euclidean n-space E, then the differential dI corresponding to the increment $h = (h_1, h_2, \cdots, h_n)$ is given by

$$dI = \sum_{\nu=1}^{n} \partial I/\partial x_\nu h_\nu = (\text{grad } I, h) \tag{2.1}$$

where the parentheses indicate the scalar product. Thus $g(x) = \text{grad } I(x)$ assigns to the point x of E the linear form $l_x(h) = dI = (g(x), h)$, that is, an element of the conjugate space E^* of E. This remark motivates the following definition.

DEFINITION 2.1. Let E be a real Banach space.[2] Let $I = I(x)$ be a scalar function defined in E or part of E. Suppose that I possesses a continuous Fréchet differential, that is, that there exists a linear

An address delivered before the Norman meeting of the Society on November 23, 1951 by invitation of the Committee to Select Hour Speakers for Western Sectional Meetings; received by the editors December 24, 1951.

[1] The use of gradients in Hilbert space for the treatment of functional equations seems to occur first in [**9**]. (Numbers in brackets refer to the bibliography at the end of the paper.)

[2] For the definition of a Banach space see, for example, [**14**, p. 10].