# CHARACTERISTIC ROOTS AND FIELD OF VALUES OF A MATRIX

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1. **Introduction.** Let $A=(a_{ij})$ be a square matrix of order $n$ whose elements are in the field of complex numbers. The complex number $\lambda$ is a *characteristic root* of the matrix $A$ if the determinant of the matrix $\lambda I-A$ is zero. It follows that $\lambda$ is a characteristic root of $A$ if, and only if, there exists a vector $x$ such that $xx^*=1$ and

$$Ax^* = \lambda x^*, \tag{1}$$

where * is used to denote transposed conjugate. By taking transposed conjugates on both sides in (1) we obtain

$$xA^* = \bar{\lambda} x. \tag{2}$$

From (1) it follows that $\lambda=xAx^*$. The set of all complex numbers $zAz^*$ where $zz^*=1$ is called the *field of values* [25][1] of the matrix $A$. It follows that *the characteristic roots of $A$ belong to the field of values of $A$.*

Beginning with Bendixson [3] in 1900, many writers have obtained limits for the characteristic roots of a matrix. In many cases these were actually limits for the field of values of the matrix [14]. In an address delivered before the Mathematical Association of America in 1938, Browne [10] gave a summary of these results up to that time. It is the purpose here to discuss some of the results obtained since the time of Browne's paper.

2. **Some well known results.** If $x$ and $y$ are two vectors such that $xx^*=yy^*=1$ and $X$ and $Y$ are unitary matrices with leading vectors $x$ and $y$ respectively, then $xX^*=yY^*$ and hence $y=xX^*Y$, or $y=xU$ where $U$ is a unitary matrix. Also, if $xx^*=1$ and $y=xU$ where $U$ is a unitary matrix, then $yy^*=xUU^*x^*=xx^*=1$. It follows, therefore, that $A$ and $UAU^*$ have the *same field of values and the same characteristic roots* for every unitary matrix $U$. It may be readily shown [21] that *the field of values of $A$ is identical with the set of all diagonal elements of the matrices $UAU^*$ where $UU^*=I$.* If $A$ is *Hermitian* there exists a unitary matrix $U$ such that $UAU^*=\text{diag.}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.