FUNCTIONS ON LOCALLY COMPACT GROUPS

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I. INTRODUCTION

1. **Background.** The subject of this address is a branch of mathematics which may be regarded as a combination of the classical theory of representations of finite groups by matrices and that part of analysis centering around the theory of Fourier series and integrals. The connection between these two apparently diverse subjects arises simply enough from the fact that the real numbers and the real numbers modulo 2π form groups under addition.

In one form of the theory of representations of a finite group G a central role is played by the so-called group ring or group algebra. This is usually defined as the set of all formal linear combinations of group elements $c_1s_1+c_2s_2+ \cdots +c_ns_n$, where each $s_i \in G$ and each c_i is a complex number. Two such expressions are added in the obvious manner and are multiplied by writing down the formal product and simplifying by means of the distributive law and the given multiplication of group elements. It may also be defined (and this is the definition we shall use) as the vector space of all complex-valued functions on G with multiplication defined by the formula

$$(f*g)(s) = \sum_{t \in G} f(st^{-1})g(t).$$

If we regard $c_1s_1+c_2s_2+ \cdots +c_ns_n$ as the function whose value at s_i is c_i, it is not difficult to see that these two definitions amount to the same thing. A basic result in the theory of group representations asserts that the group algebra is a direct sum of minimal two-sided ideals. Now it is easy to see that a linear subspace M of the group algebra is a left (right) ideal if and only if it is closed with respect to left (right) translation; that is, if and only if $f \in M$ implies that ${}_af(f_a)$ is in M for all a in G where ${}_af(s)=f(as)$ and $f_a(s)=f(sa)$. Thus every function on G is uniquely expressible as a sum of functions each of which has the property that its right and left translates generate a linear subspace which is minimal with respect to translation invariance.

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