GRILLE DECOMPOSITION AND CONVEXIFICATION THEOREMS FOR COMPACT METRIC LOCALLY CONNECTED CONTINUA

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A convex metric space is a metric space in which for each two points a and b there is a point c, different from a and b, such that $d(a, b) = d(a, c) + d(c, b)$, where d is the distance function. Menger has raised the question whether for every Peano space (that is, compact, metric, locally connected continuum) it is possible to define a distance function (preserving the original topology) with respect to which the space is convex.[1] In the present paper, this question is answered in the affirmative. For a general discussion of the problem and various partial solutions of it, see Menger[2] and Blumenthal.[3] After Blumenthal's book was written, Beer[4] and Bing[5] established convexification theorems for the one-dimensional and finite-dimensional cases respectively.

Only a short time after the present paper was written, Bing independently obtained a proof of the convexification theorem. His paper will be published in this Bulletin.

As Menger has shown[6] every compact convex metric space is locally connected. An affirmative answer to Menger's question therefore shows that for compact metric spaces, local connectedness and the existence of a convex metric are equivalent.

Theorems 1 and 2, our principal preliminary results, may be of interest independently of the use made of them here; they do not appear to be readily deducible from the convexification theorem, and their conclusions are considerably stronger than our present purposes require.

THEOREM 1. *Let S be a Peano space and let R, R', and D be open sets such that* (1) $R \subset R' \subset D$, (2) *the closures of* $\beta(R) - \beta(D)$ *and*

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[1] K. Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann. vol. 100 (1928) pp. 81 and 98.

[2] Menger, loc. cit. p. 96 ff.

[3] Leonard M. Blumenthal, *Distance geometries*, University of Missouri Studies, vol. 13, 1938, No. 2, p. 43.

[4] G. Beer, *Beweis des Satzes, dass jede im kleinen zusammenhängende Kurve konvex metrisiert werden kann*, Fund. Math. vol. 31 (1938) p. 281.

[5] R. Bing, *A convex metric for a locally connected continuum*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 812–819.

[6] Menger, loc. cit., p. 98.