A NONHOMOGENEOUS MINIMAL SET

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1. Introduction. In this note we consider the following question: *does there exist a compact minimal set which is of dimension* 0 *at some of its points and of positive dimension at others?* We answer the question in the affirmative by constructing a compact plane set X and a homeomorphism T of X onto X such that X is minimal with respect to T (that is, contains no proper closed subset Y with $T(Y) \subset Y$) and such that X possesses the desired property. As a result, there exist nonhomogeneous minimal sets.

An outline of the procedure is as follows. A compact, totally disconnected subset A of the x-axis in the plane and a homeomorphism f of A onto A are defined so that A is minimal with respect to f. Two real functions b_0 and b_1 are then defined on A with $0 \leqq b_0(x) \leqq b_1(x) \leqq 1$. We then let X be the set of all points $(x, tb_1(x)+(1-t)b_0(x))$ for $x \in A$ and $0 \leqq t \leqq 1$, thus in effect erecting a vertical interval or a point over each $x \in A$. Then T is defined so as to send the point determined by x and t into the point determined by $f(x)$ and t.

2. The example.

DEFINITIONS. Let A_i denote the set of integers $1, \cdots, 3^i$ and let π_{i+1} be the map from A_{i+1} to A_i defined by $\pi_{i+1}(p) = p \bmod 3^i$ for $p \in A_{i+1}$. Let A designate the limit space of the sequence (A_i, π_{i+1}) [1].[1] Then A is a compact totally disconnected metric space. Let f_i be the map from A_i onto A_i defined by $f_i(p) = (p+1) \bmod 3^i$; then $\pi_{i+1}f_{i+1} = f_i\pi_{i+1}$. It follows that the map defined by $f(x) = f_i((x_i))$ for $x = (x_i) \in A$ is a homeomorphism of A onto A. Moreover A is minimal with respect to f for if $x = (x_i) \in A$ and $y = (y_i) \in A$, then $f^{y_n - x_n}(x)$ has its first n coordinates equal to those of y.

Let $x = (x_i) \in A$; the points of A_{i+1} mapping onto x_i under π_{i+1} are $x_i + \alpha \cdot 3^i$, $\alpha = 0, 1, 2$. Define α_i by $x_{i+1} = x_i + \alpha_i \cdot 3^i$. We call the subsequence $\beta_1, \beta_2, \cdots$ of $\alpha_1, \alpha_2, \cdots$ consisting of all $\alpha_i \neq 1$ the *associated sequence* for x, and define several functions of x:

Let $a(x)$ be the number of elements in the associated sequence for x ($a(x)$ is either a non-negative integer or ∞).

Let $b_0(x) = 0$ *if* $a(x) = 0$, $b_0(x) = (1/2)\sum_{j=1}^{a(x)} \beta_j/2^j$ *if* $a(x) > 0$.

Let $b_1(x) = b_0(x) + \sum_{j>a(x)} 1/2^j = b_0(x) + 1/2^{a(x)}$ if $a(x) < \infty$, *and let*

Presented to the Society, September 10, 1948; received by the editors July 20, 1948.

[1] Numbers enclosed in brackets refer to the bibliography at the end of the paper.