TRANSCENDENCE OF FACTORIAL SERIES WITH PERIODIC COEFFICENTS

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It is well known that every real number α can be represented by a factorial series

$$\alpha = \frac{a_1}{1!} + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!} + \cdots, \tag{1}$$

where the a_n ($n=1, 2, \cdots$) are integers and, moreover, $0 \leqq a_n < n$ (for $n=2, 3, \cdots$). This representation is unique for the irrational numbers α, while every rational α can be represented either with almost all[1] $a_n = 0$ or with almost all $a_n = n-1$.

The representation (1) was discussed and the aforesaid properties were proved by M. Stéphanos [1].[2] But an even more general type of series had already been studied by G. Cantor [2] (not known to M. Stéphanos). These series have later been called "Cantor series" (cf. [3]).

In this note we consider the case in which the factorial series (1) has periodic coefficients a_n and we prove the following theorem:

THEOREM 1. *Every number α represented by a factorial series* (1) *with periodic coefficients is transcendental* (*except for the trivial case where almost all a_n are zero*).

The above condition $0 \leqq a_n < n$ (for $n=2, 3, \cdots$) is not used at all in the following proof. Moreover, the coefficients a_n need not necessarily be integers; the a_n can be *any algebraic numbers*. Then Theorem 1 and its proof still hold.

We generalize Theorem 1 further:

THEOREM 2. *If the power series*[3]

$$\phi(z) = \sum_{n=1}^{\infty} \frac{a_n}{n!} z^n \tag{2}$$

has algebraic coefficients a_n (not almost all of them being zero) which form a periodic sequence, then $\phi(z)$ is a transcendental number for every

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[1] The expression "almost all" is used in the sense of "all but a finite number."

[2] Numbers in brackets refer to the bibliography at the end.

[3] Under the conditions of Theorem 2, $\phi(z)$ is an entire function.