ON THE EUCLIDEAN ALGORITHM IN QUADRATIC NUMBER FIELDS

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1. **Introduction.** Let m be a square-free rational integer. The field $R(m^{1/2})$ is said to be Euclidean or that the Euclidean algorithm exists in $R(m^{1/2})$ if for integers α, $\beta \neq 0 \subset R(m^{1/2})$ there exists an integer $\gamma \subset R(m^{1/2})$ such that

$$|N(\alpha - \beta\gamma)| < |N(\beta)|.$$

The problem of determining in what fields $R(m^{1/2})$ the algorithm exists has been worked out except for m equal to a prime of the form $24n+1$ and greater than 97. In this paper it is shown that the Euclidean algorithm does not exist for $m=24n+1>97$ except possibly for $m=193$, 241, 313, 337, 457, and 601. The problem is not settled in these six cases.

2. **Previous results.** In order that a field be Euclidean the class number must be 1. However, this condition is not sufficient for, as Dedekind pointed out [1][1], the field $R(-19^{1/2})$ has class number 1 but is not Euclidean. L. E. Dickson [2] showed that for m negative the Euclidean algorithm exists only if $m=-1$, -2, -3, -7, and -11. For m positive, the algorithm has been shown to exist for the following values of m:

(1) 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73, 97.

Except for the last two values in (1) the proofs have been obtained by O. Perron [3], A. Oppenheim [4], R. Remak [5], N. Hofreiter [6], and A. Berg [7]. It was pointed out by I. Schur [4, p. 351] that the algorithm does not exist for $m=47$. A. Oppenheim [4] proved that for $m=23$ and $m=53$ the algorithm does not exist. N. Hofreiter [8] proved non-existence for $m\equiv 14 \pmod{24}$ and [6] for $m=77$ and $m\equiv 21 \pmod{24}$, $m>21$. E. Berg [7] and J. F. Keston [9] proved non-existence for $m\not\equiv 1 \pmod 4$ except for the values listed in (1). Also, apart from (1) H. Behrbohm and L. Rédei [10] showed that the algorithm can exist only in the following three cases.

I. $\quad m = p \equiv 13 \pmod{24}$,

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[1] Numbers in brackets refer to the bibliography at the end of the paper.