A THEOREM IN FINITE PROJECTIVE GEOMETRY

CHUNG-TAO YANG

It is known[1] that if F is a Galois field of order p^m and S_n^m a finite projective n-space over F, then each line of S_n^m passes through p^m+1 points and in S_n^m appear $(N_{n,0}^m N_{n,1}^m \cdots N_{n,t}^m)/(N_{t,0}^m N_{t,1}^m \cdots N_{t,t}^m)$ t-spaces S_t^m, where

$$N_{i,j}^m \equiv p^{mi} + p^{m(i-1)} + \cdots + p^{mj}.$$

In case F possesses a subfield F' of order p^r, there exists in S_n^m at least one n-subspace S_n^r, that is, a finite projective n-space, on which the points contained in a line are p^r+1 in number. The converse is also true.

The object of this note is to prove the following theorem.

In order to divide an S_n^m into several S_n^r such that one and only one S_n^r contains a given point, it is necessary and sufficient that r is a divisor of m and that m/r is relatively prime to $n+1$.

We first prove the necessity of the condition.

From the above remark, m is evidently divisible by r. By hypothesis every point of S_n^m is contained in one and only one S_n^r; we infer that $N_{n,0}^r = (p^{r(n+1)}-1)/(p^r-1)$ is a divisor of $N_{n,0}^m = (p^{m(n+1)}-1)/(p^m-1)$. Hence $(m/r, n+1)=1$ is a consequence of the following lemma.

LEMMA. *Let α, β, and $a>1$ be three natural integers;*

$$(a-1)(a^{\alpha\beta}-1)/(a^\alpha-1)(a^\beta-1)$$

is an integer if and only if $(\alpha, \beta)=1$.

To prove this we note that $(\alpha, \beta)=1$ implies $(a^\alpha-1;\ a^\beta-1)=a-1$, and both $a^\alpha-1$ and $a^\beta-1$ are divisors of $a^{\alpha\beta}-1$, so that

$$(a-1)(a^{\alpha\beta}-1)/(a^\alpha-1)(a^\beta-1)$$

is an integer.

Conversely, suppose that $(a-1)(a^{\alpha\beta}-1)/(a^\alpha-1)(a^\beta-1)$ is an integer. If on the contrary $(\alpha, \beta)>1$, on denoting a prime factor of (α, β) by q so that $\alpha=\gamma q$ and $\beta=\delta q$, we have

$$\frac{(a-1)(a^{\alpha\beta}-1)}{(a^\alpha-1)(a^\beta-1)} = \frac{(a-1)(a^{\gamma q(\delta q-1)} + a^{\gamma q(\delta q-2)} + \cdots + a^{\gamma q} + 1)}{a^{\delta q}-1}.$$

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[1] O. Veblen and W. H. Bussey, *Finite projective geometries*, Trans. Amer. Math. Soc. vol. 7 (1906) p. 244.