A PROBLEM OF ROBINSON

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We propose to prove in this paper a conjecture of R. M. Robinson which generalizes to domains of connectivity $n \geqq 3$ a result which he proved [3][1] for doubly-connected domains.

Let D be a finite domain of the z-plane bounded by $n \geqq 2$ simple, closed, disjoint analytic curves, $C_1, \cdots, C_n$, whose sum we denote by C. Let z_0 be a point of D. We consider the class Ω of functions $F(z)$ in D with at most one simple pole

$$F(z) = \frac{\alpha_F}{z - z_0} + a_0 + a_1(z - z_0) + \cdots, \tag{1}$$

at $z = z_0$, which are regular in the remainder of D and satisfy

$$\limsup_{z \to C} |F(z)| \leqq 1. \tag{2}$$

The family Ω is compact. Thus the real number

$$\sigma(\zeta_0) = \max_{F \in \Omega} |F(\zeta_0)| \tag{3}$$

is defined for every $z_0 \neq z_0$ of D.

Since the function $F(z) \equiv 1$ is in the class Ω, it is clear that $\sigma(\zeta_0) \geqq 1$ in D. The question raised by Robinson is whether or not there exists in D a nonempty set A on which $\sigma(\zeta_0) = 1$. He has solved the problem for the case of an annulus. If the annulus under consideration consists of the points z with $0 < r < |z| < 1$, and if $r < z_0 < 1$, then he has shown [3] that for $-1 < \zeta_0 < -r$ we have $\sigma(\zeta_0) = 1$, while $\sigma(\zeta_0) > 1$ in the remainder of D. Hence the set A consists here of the segment $-1 < \zeta_0 < -r$. From this result Robinson obtains a highly elegant treatment of questions of the Schwarz lemma type for an annulus.

We shall proceed in the opposite direction. We shall use results obtained in the author's thesis [2] for bounded functions to extend Robinson's theorem to domains of connectivity $n \geqq 3$.

It follows from the methods in the paper cited that there is a unique extremal function $F_0(z; \zeta_0) \in \Omega$ with

$$F_0(\zeta_0; \zeta_0) = \sigma(\zeta_0). \tag{4}$$

If $G(z; \zeta)$ denotes the Green's function of D, we have

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[1] Numbers in brackets refer to the bibliography at the end of the paper.