REAL ROOTS OF DIRICHLET L-SERIES

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Let k be a positive integer. Let χ be a real, non-principal character (mod k) and

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

be the corresponding L-series, which converges uniformly for $R(s) \geqq \epsilon > 0$. If it could be shown that uniformly in k there is no real zero of $L(s, \chi)$ for

$$s \geqq 1 - \frac{A}{\log k},$$

where A is a constant, then the existing theorems on the distribution of primes in arithmetic progressions could be greatly improved (see [1]).[1] Moreover by Hecke's Theorem (see [2]) it would follow that uniformly in k

$$L(1, \chi) > \frac{B}{\log k}$$

where B is a constant. This would be a considerable improvement over Siegel's Theorem (see [3]), and would lead to an improved lower bound for the class number of an imaginary quadratic field.

In the present paper, we shall show that for $2 \leqq k \leqq 67$, $L(s, \chi)$ has no positive real zeros. By combining this information with the results of [1], we infer very sharp estimates on the distribution of primes in arithmetic progressions of difference k for $k \leqq 67$.

The methods used for $k \leqq 67$ certainly will suffice for many other k's greater than 67. They may possibly suffice for all k, but we can find no proof of this.[2]

In [5], S. Chowla has considered the positive real zeros of $L(s, \chi)$, and shown that for many explicit k's, no positive real zeros exist. However Chowla could not settle whether his methods would suffice

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] These methods have been tried on all $k \leqq 227$ and it has been ascertained that except for the cases $k=148$ and $k=163$, $L(s, \chi)$ has no positive real zeros for $2 \leqq k \leqq 227$. Cases $k=148$ and $k=163$ are now being studied and any results obtained about them will appear in the Journal of Research of the National Bureau of Standards.