INTEGRAL DISTANCES IN BANACH SPACES

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In a recent paper,[1] Anning and Erdös have shown that in a finite-dimensional euclidean space a necessary and sufficient condition that an infinite set of points have the property that the distance between any two of them be an integer is that the points lie on a straight line. In this paper we shall investigate to what extent this theorem is true for general normed linear spaces of finite or infinite dimension. The theorem as stated is evidently not true for spaces of infinite dimension since in the space l^2 the infinite set of elements $z_i = (0, 0, \cdots, 2^{1/2}/2, 0, \cdots)$ with $2^{1/2}/2$ in the ith place and zero elsewhere have the property that $\|z_i - z_j\| = 1$ for $i \neq j$ and the z_i are linearly independent and not collinear. We shall prove that if a Banach space is not strictly convex there exists an infinite set of points each at an integral distance from the others which do not lie on a straight line. If the space is strictly convex, any such set has the property that if an infinite number of its points lie on a straight line then all the points of the set lie on the line. It is possible to prove certain theorems in analysis by applying these theorems to function spaces.

We shall define a *minimal set* connecting the points x_1 and x_2 in a Banach space X to be the set of all points of the space for which $\|x_1 - x\| + \|x_2 - x\| = \|x_1 - x_2\|$. A *straight line* containing x_1 and x_2 is the set of all points of the form $\alpha x_1 + \beta x_2$ where $\alpha + \beta = 1$. The set of points of the line for which α and β are non-negative is the *line segment* joining x_1 and x_2. If $\alpha + \beta = 1$, $\alpha \geqq 0$, $\beta \geqq 0$

$$\begin{aligned} \|x_1 - (\alpha x_1 + \beta x_2\| &+ \|x_2 - (\alpha x_1 + \beta x_2)\| \\ &= \|(1 - \alpha)x_1 - \beta x_2\| + \|(1 - \beta)x_2 - \alpha x_1\| \\ &= \|\beta x_1 - \beta x_2\| + \|\alpha x_2 - \alpha x_1\| = \beta\|x_2 - x_1\| + \alpha\|x_2 - x_1\| \\ &= (\alpha + \beta)\|x_2 - x_1\| = \|x_2 - x_1\| \end{aligned}$$

and the line segment joining x_1 and x_2 is in the minimal set determined by x_1 and x_2. A space in which the minimal set consists only of the line segment for any pair of points is called a *straight line space.* We shall show that such spaces are the strictly convex spaces.

THEOREM 1. *A necessary and sufficient condition that the minimal*

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[1] N. A. Anning and P. Erdös, *Integral distances*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 598–600.