ON A THEOREM OF H. CARTAN

RICHARD BRAUER

As an application of the Galois theory of skew fields, H. Cartan[1] obtained recently the following theorem: If K is a skew field of finite rank over its center C, the only skew fields H, $C\subseteq H\subseteq K$, which are mapped into themselves by every inner automorphism of K are K and C.

I give a very short and direct proof removing at the same time the finiteness assumption, In fact, we have:

THEOREM. *If H is a skew field contained in the skew field K, and if every inner automorphism of K maps H into itself, then H is either K, or H belongs to the center of K.*

PROOF. If $a\in K$, $b\in H$, the assumption shows that an equation

$$ba = ab_1 \tag{1}$$

with $b_1\in H$ holds (for $a=0$ this is true with $b_1=b$). Also,

$$b(1 + a) = (1 + a)b_2 \tag{2}$$

with $b_2\in H$. On subtracting (1) from (2), we find

$$b - b_2 = a(b_2 - b_1).$$

If a does not lie in H, this implies $b_2=b_1$ and hence $b=b_2$. Then $b_1=b$, that is, $ba=ab$. Every element a of K which does not belong to H commutes therefore with every element of H.

Suppose that H does not belong to the center of K. There exists an element b of H and an element c of K such that

$$bc \neq cb. \tag{3}$$

The remark above shows that $c\in H$. If $H\neq K$, there exist elements a outside of H in K. Then $a+c$ does not belong to H either. Hence $a+c$ and a both commute with $b\in H$,

$$b(a + c) = (a + c)b, \ ba = ab.$$

These two equations are not consistent with (3), and the theorem is proved.

The same argument applies under much weaker assumptions. For instance, it is sufficient to assume that K is a (not necessarily associa-

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[1] C. R. Acad. Sci. Paris vol. 224 (1947) pp. 249–251.