# SOME REMARKS ABOUT LIE GROUPS TRANSITIVE ON SPHERES AND TORI

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The present note pertains principally to two papers of D. Montgomery and H. Samelson [**1**, **2**],[1] in which the authors study compact Lie groups transitive on tori [**1**] and spheres [**2**]. I will here prove in another way, generalize, and sharpen a part of their results. §1 contains the remarks to [**1**], §2 to [**2**]; they are independent of one another and the methods used in both are quite different.

I recall first the definition and some simple properties of homogeneous spaces. A manifold $W$ is a homogeneous space under the Lie group[2] $G$ if to each element $a$ of $G$ there corresponds a differentiable transformation $T_a: x \to T_a(x)$ of $W$ into itself such that:

(1) $T_a(x)$ depends continuously on the pair $a \in G$, $x \in W$.

(2) To the product $(ab)$ corresponds the mapping $x \to T_{(ab)}(x) = T_a[T_b(x)]$.

(3) Given any two points $x$, $y$ in $W$, there exists $a \in G$ such that $T_a(x) = y$ (that is, $G$ is *transitive* on $W$).

$G$ is said to be *effective* on $W$ if only the identity element $e$ of $G$ induces the identity transformation of $W$.

Let us choose an arbitrary point $x$ of $W$. The set of elements $h$ in $G$ for which $T_h(x) = x$ is a closed subgroup $H$ of $G$, called the *associated group*. As is well known [**3**, no. 29], $W$ may be identified with the space of left cosets $G/H$, the mappings $T_a$ being then: $xH \to (ax)H$. Actually, $H$ depends on the choice of $x \in W$ and should be denoted $H_x$, but I shall in general drop the index $x$ as there will be no danger of confusion and also because all the groups $H_x$ ($x \in W$) are conjugate to each other in $G$.

When considering a homogeneous space as the space of left cosets, it is quite easy to prove that *every subgroup of $H$ which is invariant in $G$ induces the identity mapping of $W$*, and, conversely, *a subgroup of $H$, each element of which induces the identity of $W$, is invariant in $G$.*

## 1. The $n$-dimensional torus as a homogeneous space.

In [**1**], D. Montgomery and H. Samelson proved that a Lie group which acts transitively and effectively on the $n$-dimensional torus is itself the $n$-dimensional toral group $T^n$. Actually, as they remark at the end of

Received by the editors June 8, 1948.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] The manifolds and Lie groups considered here are always *compact*.