REMARKS ON A PAPER BY ZEEV NEHARI

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In the preceding paper Zeev Nehari has proved some interesting inequalities for the Schwarzian derivative of a univalent (=schlicht) function. Thus *if* $f(z)$ *is univalent in the unit circle, then*

$$(1) \qquad |\{f(z), z\}| \leqq 6[1 - |z|^2]^{-2}$$

while if

$$(2) \qquad |\{f(z), z\}| \leqq 2[1 - |z|^2]^{-2},$$

then $f(z)$ *is univalent for* $|z| < 1$. The object of the present note is to show that 2 *is the best possible constant* in (2) in the following sense:

For every $C > 2$ *there exists a function* $f(z)$ *such that for* $|z| < 1$ *we have* (i) $f(z)$ *is holomorphic,* (ii) $f(z)$ *takes on the value one infinitely often, and* (iii) $|\{f(z), z\}| \leqq C[1 - |z|^2]^{-2}$ *with equality for real values of* z.

An explicit example of such a function is given by

$$(3) \qquad f(z) = \left(\frac{1 - z}{1 + z}\right)^{\gamma i},$$

where γ is a real constant, $f(0) = 1$, and $C = 2(1 + \gamma^2)$.

In view of the background of the problem, the following approach is natural. Let F_a denote the family of fractional linear transforms with constant coefficients of the quotient of two linearly independent solutions of the differential equation,

$$(4) \qquad \frac{d^2y}{dz^2} + a(1 - z^2)^{-2}y = 0,$$

where a is an arbitrary parameter. If $f(z) \in F_a$, then

$$\{f(z), z\} = 2a(1 - z^2)^{-2}.$$

If one function $f(z)$ of F_a is univalent in the unit circle, then they all are. Let us determine the region U of the a-plane such that if $a \in U$, then the functions of F_a are univalent for $|z| < 1$. By Nehari's Theorem I the region $|a| \leqq 1$ belongs to U and (1) makes it plausible that U is contained in $|a| \leqq 3$.

Equation (4) has elementary solutions. The indicial equations at

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