COMPOSITION OF BINARY QUADRATIC FORMS

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1. **Introduction.** The composition of quadratic forms, as originated by Gauss,[1] is based on bilinear transformations. Thus, if a quadratic form $f_1 = \sum a_{ij}x_ix_j$ is expressible as a product of two forms $f_2(y_1, \cdots, y_n)$ and $f_3(z_1, \cdots, z_n)$ by means of a bilinear substitution $x_\alpha = \sum a_{\alpha\beta\gamma}y_\beta z_\gamma$, and if the determinants of order n in the n-by-n^2 matrix $(a_{\alpha\beta\gamma})$ are relative prime, f_1 is called the compound, or product under composition, of f_2 and f_3. There are few examples of composition except for quadratic forms, and there it is confined to certain classes of forms in two, four, and eight variables.

Now there is evidence that quadratic forms not admitting composition have certain properties akin to those which are most easily established in the case of binaries by use of composition. This suggests that the use of bilinear transformations is too restrictive, and that other useful definitions of composition may be possible. Dirichlet[2] did in fact base a theory of composition of binary quadratic forms on the representation of numbers. However, bilinear transformations appear (loc. cit., p. 159, formula (5)) in his proof of the uniqueness of the product class. Again, Brandt[3] gave a theory of composition for binaries, based on integral linear transformations of a Grundform into multiples of the binary quadratic forms of a given discriminant. The extension of this to n variables appears to be difficult.

In this article we define a compound of binary quadratic forms in a manner basically related to that of Dirichlet; and prove the uniqueness of the product class without using bilinear transformations. We also show that the basic lemma (due to Gauss) can be extended to quadratic forms in n variables. All the usual consequences of composition of binary quadratic forms can be derived from our present approach, some of them more simply. But we shall not enter into these details here.

2. **Gauss's lemma and its generalization.** The basic lemma of

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[1] C. F. Gauss, *Disquisitiones Arithmeticae*, 1801, Articles 235–249 et. seq.

[2] G. L. Dirichlet, *De formarum binarum secundi gradus compositione*, Berlin, 1851. Reprinted in Journal für Mathematik vol. 47 (1854) pp. 155–160; Werke, II, 1897, pp. 105–114. French translation, Journal de Mathematik (2) vol. 4 (1859) pp. 389–398. Also, Dirichlet-Dedekind, *Zahlentheorie*, Supplement X, §§145–9, 1871, 1879, 1894.

[3] H. Brandt, Journal für Mathematik vol. 150 (1919) pp. 1–46.