THE ASYMPTOTIC DISTRIBUTION OF THE SUM OF A RANDOM NUMBER OF RANDOM VARIABLES

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1. Introduction. If a random variable (r. v.) Y is the sum of a large but constant number N of independent components

$$Y = X_1 + \cdots + X_N, \tag{1}$$

then under appropriate conditions on the X_j it follows from the central limit theorem that the distribution of Y will be nearly normal. In many cases of practical importance, however, the number N is itself a r. v., and when this is so the situation is more complex.

We shall consider the case in which the X_j $(j=1, 2, \cdots)$ are independent r. v.'s with the same distribution function (d. f.) $F(x) = P[X_j \leqq x]$, and in which the non-negative integer-valued r. v. N is independent of the X_j. The d. f. of N we shall assume to depend on a parameter λ, so that the d. f. of Y is a function of λ which may have an asymptotic expression as $\lambda \rightarrow \infty$. In the degenerate case in which for any integer λ, N is certain to have the value λ, the problem reduces to the ordinary central limit problem for equi-distributed components.

In the general case the d. f. of N for any λ is determined by the values $\omega_k = P[N=k]$ $(k=0, 1, \cdots)$, where the ω_k are functions of λ such that for all λ,

$$\omega_k \geqq 0, \qquad \sum_0^\infty \omega_k = 1.$$

We shall use Greek letters to denote functions of the parameter λ; in particular we define

$$\begin{aligned} \alpha &= E(N) = \sum_0^\infty \omega_k \cdot k, \\ \beta^2 &= E(N^2) = \sum_0^\infty \omega_k \cdot k^2 \qquad \text{(assumed finite for all } \lambda\text{)}, \\ \gamma^2 &= \operatorname{Var}(N) = \sum_0^\infty \omega_k \cdot (k-\alpha)^2 = \beta^2 - \alpha^2, \\ \theta(t) &= E(e^{i(N-\alpha)t/\gamma}) = \sum_0^\infty \omega_k \cdot e^{i(k-\alpha)t/\gamma}, \end{aligned} \tag{2}$$

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