NOTE ON A THEOREM DUE TO BORSUK

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1. Introduction. Let A, $B\subset A$ and B' be compacta, which are[1] ANR's (absolute neighbourhood retracts). Let $B'\subset A'$ where A' is a compactum, and let $f:(A, B)\to(A', B')$ be a map such that $f|(A-B)$ is a homeomorphism onto $A'-B'$. Thus A' is homeomorphic to the space defined in terms of A, B, B' and the map $g=f|B$ by identifying each point $b\in B$ with $gb\in B'$. K. Borsuk [3] has shown that A' is locally contractible. It is therefore an ANR if $\dim A'<\infty$. The main purpose of this note is to prove, without this restriction on $\dim A'$:

THEOREM 1. *A' is an ANR.*

We also derive some simple consequences of this theorem. For example, it follows that the homotopy extension theorem, in the form in which the image space is arbitrary, may be extended[2] from maps of polyhedra to maps of compact ANR's, P and $Q\subset P$. That is to say, if $f_0:P\to X$ is a given map, the space X being arbitrary, and if $g_t:Q\to X$ is a deformation of $g_0=f_0|Q$, then there is a homotopy $f_t:P\to X$, such that $f_t|Q=g_t$. For let $R=(P\times 0)\cup(Q\times I)\subset P\times I$ and let $h:R\to X$ be given by $h(p, 0)=f_0p$, $h(q, t)=g_tq$ $(p\in P,\ q\in Q)$. Since $Q\times I$ is (obviously) a compact ANR it follows from Theorem 1, with $A=Q\times I$, $B=Q\times 0$, $B'=P\times 0$, $A'=R$ that R is an ANR. Therefore R is a retract of some open set $U\subset P\times I$. If $\theta: U\to R$ is a retraction, then $h\theta: U\to X$ is an extension of $h:R\to X$ throughout U. This is all we need for the homotopy extension theorem (see [5, pp. 86, 87]). Thus we have the corollary:

COROLLARY. *A given homotopy, $g_t:Q\to X$, of $g_0=f_0|Q$, can be extended to a homotopy, $f_t:P\to X$, where P and $Q\subset P$ are compact ANR's and $f_0:P\to X$ is a given map of P in an arbitrary space X.*

We also use Theorem 1 to prove another theorem. We shall describe a map $\xi:X\to Y$ as a *homotopy equivalence* if, and only if, there is a map, $\eta: Y\to X$, such that $\eta\xi\simeq 1$, $\xi\eta\simeq 1$, where X and Y are any two spaces. Thus the statement that $\xi:X\to Y$ is a homotopy equivalence implies that X and Y are of the same homotopy type. Let A, B, A', B' and $f:(A, B)\to(A, B')$ be as in Theorem 1 and let $g=f|B$.

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[1] For an account of these spaces, on which this note is based, see [2]. Numbers in brackets refer to the references cited at the end of the paper.

[2] Cf. [4].