A NONASSOCIATIVE METHOD FOR ASSOCIATIVE ALGEBRAS

SAUNDERS MACLANE[1]

This note exhibits a nonassociative proof for a strictly associative theorem concerned with the "Galois theory" of associative crossed product algebras. The theorem in question has also been established by somewhat more elaborate associative computations: it is perhaps of interest that the nonassociative proof to be given here appears to be both shorter and more conceptual than the associative proof. Practically no technical facts about nonassociative algebras are required for our proof.

Let $K \supset N \supset P$ be fields such that both K and N are finite, separable, and normal extensions of the base field P. The Galois group of K over P, or briefly $\mathcal{G}(K/P)$, will be designated as G, and similarly,

$$\mathcal{G}(K/P) = G, \qquad \mathcal{G}(K/N) = S, \qquad \mathcal{G}(N/P) = Q. \tag{1}$$

Then $S \subset G$. Each $\alpha \in G$ is an automorphism α: $k \leftrightarrow \alpha \cdot k$ of K, and induces an automorphism $\alpha' \in Q$ of N/P; this correspondence $\alpha \rightarrow \alpha'$ provides the natural isomorphism $G/S \cong Q$. We consider functions $h(\alpha, \beta)$ with arguments α, β in G and nonzero values in the field K. The coboundary δh is a similar function of three arguments in G, defined as

$$\delta h(\alpha, \beta, \gamma) = [\alpha \cdot h(\beta, \gamma)] h(\alpha, \beta\gamma)[h(\alpha\beta, \gamma) h(\alpha, \beta)]^{-1}. \tag{2}$$

It is convenient to assume that any such function h is "normalized," in the sense that $h(I, \beta) = h(\alpha, I) = 1$, where I denotes the identity automorphism. The coboundary δh is then also normalized, for it follows that $\delta h(I, \beta, \gamma) = \delta h(\alpha, I, \gamma) = \delta h(\alpha, \beta, I) = 1$.

A factor set f of S in the multiplicative group of K is a (normalized) function $f(\sigma, \tau) \in K$ defined for arguments $\sigma, \tau \in S$ and satisfying the identity $\delta f(\rho, \sigma, \tau) = 1$, for all ρ, σ, τ in S. Each factor set f leads to a crossed product (cf. [1, Chap. V][2]) $A = (K, f)$, which is a simple algebra with center N, and which may be represented in terms of elements $u(\sigma)$, one for each $\sigma \in S$, as the set of all sums $a = \sum_\sigma k(\sigma) u(\sigma)$ with arbitrary coefficients $k(\sigma) \in K$ and with the multiplication table

$$u(\sigma) u(\tau) = f(\sigma, \tau) u(\sigma\tau), \qquad u(\sigma) k = [\sigma \cdot k] u(\sigma), \tag{3}$$

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[1] John Simon Guggenheim Memorial Fellow.

[2] Numbers in brackets refer to the bibliography at the end of the paper.