GENERALIZATION OF MENGER'S RESULT ON THE STRUCTURE OF LOGICAL FORMULAS

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Menger's paper[2] gives necessary and sufficient conditions that an expression containing sentential variables and unary and binary sentential connectives be a formula in the Łukasiewicz notation. This paper extends his result to expressions containing n-ary symbols for all $n \geqq 0$. Sentential variables and constants are treated as the case $n = 0$.

An expression is a sequence $s_1 \cdots s_k$ such that s_i for $i = 1, \cdots, k$ is an n-ary symbol for some n. An initial segment of such an expression is an expression $s_1 \cdots s_i$ where $i < k$; a terminal segment is an expression $s_t \cdots s_k$ where $t > 1$. A formula is a sequence $sz_1 \cdots z_n$ where s is an n-ary symbol, and $z_1, \cdots, z_n$ are formulas. The measure $[s]$ of an n-ary symbol s is $n-1$. The measure $[s_1 \cdots s_k]$ of an expression $s_1 \cdots s_k$ is $[s_1] + \cdots + [s_k]$.

THEOREM. *Necessary and sufficient conditions that an expression* $x = s_1 \cdots s_k$ *be a formula are:*

$$[y] \geqq 0 \textit{ for each initial segment } y \textit{ of } x, \tag{1}$$

and

$$[x] = -1. \tag{2}$$

PROOF. Suppose s is an n-ary symbol, $z_1, \cdots, z_h$, $h \geqq 0$ are formulas, z is an initial segment of a formula z_{h+1}, and $z_1, \cdots, z_{h+1}$ satisfy (1) and (2). Then

$$\begin{aligned} [sz_1 \cdots z_h] &= [s] + [z_1] + \cdots + [z_h] \\ &= (n-1) - 1 - \cdots - 1 = n - 1 - h \end{aligned} \tag{3}$$

and

$$[sz_1 \cdots z_h z] = n - 1 - h + [z] \geqq n - 1 - h. \tag{4}$$

PROOF OF NECESSITY. Let $x = sz_1 \cdots z_n$ be a formula where by the induction hypothesis s is an n-ary symbol and $z_1, \cdots, z_n$ are formulas

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[2] Menger, Karl, *Eine elementare Bemerkung über die Struktur logischer Formeln*, Ergebnisse eines mathematischen Kolloquiums, vol. 3, 1930–1931, pp. 22–23.