THE REMAINDER IN APPROXIMATIONS BY MOVING AVERAGES

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1. Introduction. Many of the processes of interpolation or smoothing are of the following sort. A function $L(s)$, defined for all real s, characterizes the process. Given a function $x(s)$, the function

$$y(t) = \sum_{j=-\infty}^{\infty} x(j)L(t-j) \tag{1}$$

is constructed, when possible; $y(t)$ is thought of as an approximation of $x(t)$. The remainder in the approximation is

$$R[x] = x(t) - y(t). \tag{2}$$

In the conventional processes of smoothing or interpolation, $L(s)$ is a function which vanishes for all $|s|$ sufficiently large. I. J. Schoenberg[2] has recently introduced a class of formulas (1), (2) in which $L(s)$ is an analytic function and the series (1) does not consist of a finite number of terms.

Schoenberg gives an elegant criterion for recognizing cases in which the approximating process is exact for polynomials of degree $n-1$; that is, cases in which $R[x]=0$, for all t, whenever $x(s)$ is a polynomial of degree $n-1$.[3] In the present paper we obtain an integral representation of such operations $R[x]$ in terms of the nth derivative $x^{(n)}(s)$. The representation is precisely of the sort that holds when $R[x]$ is a linear functional on certain spaces of functions $x(s)$ defined on a *finite* s-interval.

2. The integral representation. We shall consider an operation which is more general than (1), (2). *Let $g(s, t)$ be a function which, for each number t in a given set $\mathfrak{T}$, is of bounded variation in s on each finite s-interval.* Given any function $x(s)$, put

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[2] *Contributions to the problem of approximation of equidistant data by analytic functions*, Quarterly of Applied Mathematics vol. 4 (1946) pp. 45–99 and 112–141.

[3] Loc. cit. Theorem 2B, p. 64. Schoenberg's criterion is valid whether $L(s)$ is a symmetric function or not.

Throughout the present paper "polynomial of degree k" is to be understood as "polynomial of proper degree k or less."