ONE-TO-ONE MAPPINGS OF RINGS AND LATTICES

C. E. RICKART

1. **Introduction.** Let $\mathfrak{R}$ and $\mathfrak{S}$ be two arbitrary rings and let $\phi(r)$ denote a one-to-one, multiplicative mapping of $\mathfrak{R}$ onto $\mathfrak{S}$. The major portion of this paper is devoted to the proofs of two theorems which state that, under certain conditions on $\mathfrak{R}$, the mapping $\phi(r)$ is automatically additive. In Theorem I (§2), $\mathfrak{R}$ is an arbitrary Boolean ring and in Theorem II (§3), $\mathfrak{R}$ is any ring which contains a family $\mathcal{J}$ of minimal right ideals satisfying the following two conditions: (i) $Rr=(0)$, for every $R\in\mathcal{J}$, implies $r=0$ and (ii) each $R\in\mathcal{J}$ is of dimension greater than one over the division ring of all endomorphisms of R which commute with each endomorphism induced in R through right multiplication by elements of $\mathfrak{R}$. In §4 it is proved (Theorem III) that a one-to-one, meet preserving mapping of a distributive lattice onto a distributive lattice is necessarily join preserving.

The class of rings considered in Theorem II contains as a special case the ring of all bounded linear operators on a Banach space of dimension greater than one. Theorem II, with the additional hypothesis that the mapping be continuous, was proved in this special case by Eidelheit[1] [2].[2] The finite-dimensional case of the Eidelheit theorem was obtained by Nagumo [5].

This paper has been greatly influenced by conversations which we have had with B. J. Pettis. In particular, our interest in the questions considered here was stimulated by Pettis' conjecture that the Eidelheit theorem mentioned above could be obtained without the continuity hypothesis.

2. **Boolean rings.** By a Boolean ring [6] we mean a ring each of whose elements satisfies the conditions $r^2=r$. It is not difficult to show that a Boolean ring is necessarily commutative and that each of its elements also satisfies the condition $r+r=0$.

THEOREM I. *Any one-to-one, multiplicative mapping of a Boolean ring $\mathfrak{B}$ onto an arbitrary ring $\mathfrak{S}$ is necessarily additive.*

Let $\phi(r)$ denote the one-to-one, multiplicative mapping of $\mathfrak{B}$ onto

Presented to the Society, October 25, 1947; received by the editors September 20, 1947.

[1] Eidelheit, in fact, assumed $\mathfrak{S}$, as well as $\mathfrak{R}$, to consist of all bounded linear operators on a Banach space. He also assumed the Banach spaces to be real.

[2] Numbers in brackets refer to the bibliography at the end of the paper.