# THE CRITICAL NUMBERS FOR UNSYMMETRICAL APPROXIMATION

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**1. Statement of the main theorem.** If $\xi$ is an irrational number, then the modulus of approximability from the right, $M^+(\xi)$, is defined as the least upper bound of the values of $\mu$ for which the inequality

$$0 < \frac{A}{B} - \xi < \frac{1}{\mu B^2}$$

has infinitely many solutions. In a similar way, $M^-(\xi)$ is defined measuring the approximability of $\xi$ from the left.

The number $\xi$ is called *critical* if there is no other irrational number $\xi'$ for which

$$M^+(\xi') < M^+(\xi), \qquad M^-(\xi') < M^-(\xi).$$

That is, roughly speaking, $\xi$ is called critical if there is no other number which is harder to approximate both from the right and from the left. The purpose of this paper is to give a necessary and sufficient condition that $\xi$ be critical.

A few definitions are necessary before stating the main theorem. A sequence of non-negative integers $r_1, r_2, r_3, r_4, \cdots$ will be called *derivable* if

$$\liminf r_n + 1 = \limsup r_n < \infty,$$

that is, if ultimately just two different numbers occur in the sequence, and these are consecutive integers $k$ and $k+1$. In this case, the sequence has the form

$$r_1, r_2, \cdots, r_v, k, (k+1)^{s_1}, k, (k+1)^{s_2}, k, (k+1)^{s_3}, k, \cdots,$$

the exponents denoting repetition of the term $k+1$. Here the $s_n$ are non-negative integers; if there are consecutive $k$'s in the sequence, then we must take some of the $s_n$ equal to zero.

We shall call $s_1, s_2, s_3, \cdots$ the derived sequence. Together with $k$, this derived sequence determines the end of the primitive sequence:

$$k, (k+1)^{s_1}, k, (k+1)^{s_2}, k, (k+1)^{s_3}, k, \cdots.$$

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