PROOF OF A FORMULA OF LIOUVILLE

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1. Introduction. In his summary of the formulas stated without proof in Liouville's famous series of articles on the theory of numbers, Dickson [1][1] remarked that no proof had been published for (Q) of the Sixth Article [2]. This formula, however, was later derived by Bell [3], who obtained it by paraphrase, a general method which he applied to various identities between elliptic, abelian, and theta functions to obtain some of Liouville's formulas and others of new kinds.

Bell noted that among all of Liouville's, formula (Q) is unique, as "it is the only one which depends immediately upon the addition theorems for the elliptic functions." For this reason, together with the fact that it is the only elementary proof so far published, the derivation of formula (Q) presented below may be of interest.

2. Notation. Throughout this paper $(d_1, \delta_1, d_2, \delta_2, d_3, \delta_3)$ and $(d_1', \delta_1', d_2', \delta_2', d_3', \delta_3')$ denote integer solutions for n, a fixed positive integer, in the forms

$$n = d_1\delta_1 + d_2\delta_2 + d_3\delta_3, \tag{1}$$

$$n = d_1'\delta_1' + d_2'\delta_2' + d_3'\delta_3' \tag{2}$$

respectively, each of the d_i, δ_i, d_i', δ_i' $(i=1, 2, 3)$ being positive unless otherwise specified. Further, $F(x, y)$ denotes a function which takes a single definite value for each integral pair (x, y), satisfies the conditions

$$\begin{aligned} F(-x, y) &= -F(x, y) = F(x, -y) = F(y, x), \\ F(0, y) &= 0 = F(x, 0), \end{aligned} \tag{3}$$

but is otherwise general in the widest sense.

For the sake of brevity, certain expressions are denoted by single letters with subscripts. These are as follows:

$$\begin{aligned} S_1 = {} & d_3\{F(d_1 + d_2 + d_3, d_1 + d_2) - F(d_1 - d_2 - d_3, d_1 - d_2)\} \\ & + \delta_2\{F(\delta_1 - \delta_3, \delta_1 + \delta_2 - \delta_3) - F(\delta_1 + \delta_3, \delta_1 - \delta_2 + \delta_3)\}, \\ S_2 = {} & (d_3 - d_2)\{F(d_1 + d_3, d_1 + d_2) - F(d_1 - d_3, d_1 - d_2)\} \\ & + (\delta_2 + \delta_3)\{F(\delta_1 - \delta_3, \delta_1 + \delta_2) - F(\delta_1 + \delta_3, \delta_1 - \delta_2)\}, \end{aligned}$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.