ON THE DISTRIBUTION OF THE MAXIMUM OF SUCCESSIVE CUMULATIVE SUMS OF INDEPENDENTLY BUT NOT IDENTICALLY DISTRIBUTED CHANCE VARIABLES

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1. **Introduction.** Let $X_1, X_2, \cdots$, and so on be a sequence of chance variables and let S_i denote the sum of the first i X's, that is,

$$S_i = X_1 + \cdots + X_i \quad (i = 1, 2, \cdots, \text{ad inf}). \tag{1.1}$$

Let M_N denote the maximum of the first N cumulative sums $S_1, \cdots, S_N$, that is,

$$M_N = \max(S_1, \cdots, S_N). \tag{1.2}$$

The distribution of M_N, in particular the limiting distribution of a suitably normalized form of M_N, has been studied by Erdös and Kac [1][1] and by the author [2] in the special case when the X's are independently distributed with identical distributions.

In this note we shall be concerned with the distribution of M_N when the X's are independent but not necessarily identically distributed. In particular, the mean and variance of X_i may be any functions of i.

In §2 lower and upper limits for M_N are obtained which yield particularly simple limits for the distribution of M_N when the X's are symmetrically distributed around zero.

In §3 the special case is considered when X_i can take only the values 1 and -1 but the probability p_i that $X_i = 1$ may be any function of i. The exact probability distribution of M_N for this case is derived and expressed as the first row of a product of N matrices.

The limiting distribution of $M_N/N^{1/2}$ is treated in §4. Since the interesting limiting case arises when the mean of X_i ($i \leqq N$) is not only a function of i but also a function of N, we have to introduce a double sequence of chance variables. That is, for any N we consider a sequence of N chance variables $X_{N1}, \cdots, X_{NN}$. Let μ_{Ni} denote the mean and σ_{Ni} the standard deviation of X_{Ni}. Let, furthermore, S_{Ni} denote the sum $X_{N1} + \cdots + X_{Ni}$ and M_N the maximum of $S_{N1}, \cdots, S_{NN}$. With the help of a method used by Erdös and Kac [1], the following theorem is established in §4:

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[1] Numbers in brackets refer to the references cited at the end of the paper.