A HOMOGENEOUS DIFFERENTIAL SYSTEM OF INFINITE ORDER WITH NONVANISHING SOLUTION

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1. Introduction. The only solution of the homogeneous differential system

$$\frac{d^{2k-1}}{dt^{2k-1}}\,[t^k f(t)] = 0, \tag{1}$$

$$f(0+) = f(+\infty) = 0 \tag{2}$$

is $f(t) \equiv 0$. This may be seen by observing that none of the fundamental solutions of equation (1), t^n $(n = -k,\ -k+1, \cdots, k-2)$, satisfies the boundary conditions (2). We now replace equation (1) by another of infinite order, retaining the boundary conditions (2), and investigate the possibility of a nonvanishing solution. We let k become infinite in equation (1) after introduction of a factor. Set

$$L_{k,t}[f(x)] = \frac{(-t)^{k-1}}{k!(k-2)!}\,\frac{d^{2k-1}}{dt^{2k-1}}\,[t^k f(t)] \tag{3}$$

and consider the system

$$\lim_{k\to\infty} L_{k,t}[f(x)] = 0, \qquad 0 < t < \infty, \tag{4}$$

with the boundary conditions (2). Since the differential operator (3) serves to invert the Stieltjes transform,[1]

$$f(x) = \int_0^\infty \frac{\phi(t)}{x+t}\,dt, \tag{5}$$

$$\lim_{k\to\infty} L_{k,t}[f(x)] = \phi(t),$$

it is clear that the only solution of the system (4), (2) of the form (5) must vanish identically. Accordingly if the limit (4) exists boundedly,[2] for example, there is no nonvanishing solution of the system. However, we shall show that if the limit (4) merely exists at each point of the interval then there are many nonvanishing solutions of the system. One very simple solution is $f(t) = t(1+t)^{-2}$. We shall find

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[1] See *The Laplace transform* by D. V. Widder, Princeton University Press, 1946, p. 345.

[2] D. V. Widder, loc. cit. p. 373.