AN EXTENSION OF ALEXANDROFF'S MAPPING THEOREM

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1. Introduction. Alexandroff's fundamental mapping theorem (Überführungssatz) is a basic tool of combinatorial topology. By its use, many mapping theorems for rather general topological spaces can be shown to be consequences of the corresponding theorems for polytopes. It is especially useful in proving imbedding theorems and approximation theorems.

The chief purpose of this paper is to determine the precise conditions under which this fundamental theorem holds. It will be shown that the theorem holds in full generality, that is, for all coverings, if and only if the space is both paracompact and normal. If the space is normal, the theorem holds for all coverings which have locally finite refinements and for no others.

2. Terminology. The mapping theorem of Alexandroff concerns mappings of a space into the nerve of a covering.

By a *mapping* we mean a continuous transformation. By a *space* we mean a topological space, in general not satisfying any separation axiom. By a *covering* we mean a covering of the space by a finite or infinite collection of open sets.

The *nerve* of a covering $\mathfrak{U}$: $\{U_\alpha\}$ is a simplicial polytope, with vertices u_α in 1–1 correspondence with the nonempty sets U_α of the covering, such that $u_\alpha, u_\beta, \cdots, u_\gamma$ are vertices of a simplex of the nerve if and only if the corresponding sets $U_\alpha, U_\beta, \cdots, U_\gamma$ have a common point. We assume that the nerve is realized as a topological space in one of the following ways. The *natural nerve* $N(\mathfrak{U})$ is the nerve realized with the natural metric: $\rho(x, y) = (\sum(x_\alpha - y_\alpha)^2)^{1/2}$, where x_α, y_α are barycentric coordinates of x and y. The *geometric nerve* $G(\mathfrak{U})$ is the nerve realized with the geometric topology of Lefschetz [10, p. 9]:[1] the stars of the vertices of repeated regular subdivisions form a basis for the open sets of $G(\mathfrak{U})$. It is known [12] that $G(\mathfrak{U})$ is a metrizable space. The natural and geometric topologies coincide if and only if the nerve is locally of finite dimension [11, footnote 4].

Following Dieudonné, we call a covering $\mathfrak{U}$ of a space R *locally finite*[2]

Presented to the Society, November 2, 1946; received by the editors July 8, 1947.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] Locally finite = neighborhood-finite. "Locally finite" has been used by Lefschetz to mean star-finite.