ON SOME NEW QUESTIONS ON THE DISTRIBUTION OF PRIME NUMBERS

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1. **Introduction.** In connection with some recent unpublished investigations concerning the Riemann hypothesis one of us raised the question whether log p_n is convex for sufficiently large n, or at least whether it has few points of inflexion. (Throughout this paper $p_1=2$, $p_2=3, \cdots, p_n, \cdots$ denotes the sequence of primes.) In other words: Is it true that the inequalities

$$(1) \qquad p_{n-1}\cdot p_{n+1} > p_n^2, \qquad p_{m+1}p_{m-1} < p_m^2$$

both have infinitely many solutions? We shall show that the answer is affirmative.

A still simpler question is whether the sequence of primes itself is convex or concave from a certain n on. We shall prove that this is not so, that is, the equations

$$(2) \qquad \frac{p_{n-1}+p_{n+1}}{2} > p_n, \qquad \frac{p_{m-1}+p_{m+1}}{2} < p_m$$

have infinitely many solutions.[1]

If the well known hypothesis about prime twins is true, that is, if the equation $p_{n+1}-p_n=2$ has infinitely many solutions, (1) and (2) of course are trivially satisfied.

The first inequality of (2) is inserted only for the sake of completeness. It follows from the well known fact that lim sup $(p_{n+1}-p_n)=\infty$ (since $n!+2$, $n!+3, \cdots, n!+n$ are all composite). The proof of the other inequalities will be simple, but less trivial.

Clearly $p_{n-1}p_{n+1}>p_n^2$ implies $(p_{n-1}+p_{n+1})/2>p_n$ and $p_m>(p_{m-1}+p_{m+1})/2$ implies $p_m^2>p_{m-1}p_{m+1}$. The well known relations between the various mean values suggest the following questions: Is it true that for every t the inequalities

$$(3) \qquad \left(\frac{p_{n-1}^t+p_{n+1}^t}{2}\right)^{1/t} > p_n$$

and

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[1] Professor G. Pólya and Mr. P. Ungár communicated to us subsequently a proof very similar to our own.