ON RINGS OF ANALYTIC FUNCTIONS

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Let D be a domain in the complex plane (Riemann sphere) and $R(D)$ the totality of one-valued regular analytic functions defined in D. With the usual definitions of addition and multiplication $R(D)$ becomes a commutative ring (in fact, a domain of integrity). A one-to-one conformal transformation $\zeta = \phi(z)$ of D onto a domain Δ induces an isomorphism $f \rightarrow f^*$ between $R(D)$ and $R(\Delta)$: $f(z) = f^*[\phi(z)]$. An anti-conformal transformation

$$\zeta = \overline{\phi(z)}$$

also induces an isomorphism:

$$\overline{f(\bar{z})} = f^*[\overline{\phi(\bar{z})}].$$

The purpose of this note is to prove the converses of these statements.

THEOREM I. *If $R(D)$ is isomorphic to $R(\Delta)$, then there exists either a conformal or an anti-conformal transformation which maps D onto Δ.*[1]

THEOREM II. *If D and Δ possess boundary points, then every isomorphism between $R(D)$ and $R(\Delta)$ is induced by a conformal or an anti-conformal transformation of D onto Δ.*

Theorem I may be regarded as a complex variable analogue of theorems characterizing a topological space in terms of the family of its continuous functions. If $R(D)$ is made into a topological ring by defining $f_n \rightarrow f$ to mean that $f_n(z) \rightarrow f(z)$ uniformly in every bounded closed subset of D, then Theorem II implies that, except for a trivial special case, every isomorphism between $R(D)$ and $R(\Delta)$ is of necessity a homeomorphism.

To prove the theorems we consider a fixed isomorphism between $R(D)$ and $R(\Delta)$. It takes a function $f(z)$, $z \in D$, into a function $f^*(\zeta)$, $\zeta \in \Delta$, a set $S \subset R(D)$ into a set $S^* \subset R(\Delta)$. Let c be a complex constant.

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[1] After this paper was completed the author learned about a closely related unpublished result which was obtained by C. Chevalley and S. Kakutani several years ago. Chevalley and Kakutani proved that if to each boundary point W of B there exists a bounded analytic function defined in B and possessing at W a singularity then B is determined (modulo a conformal transformation) by the ring of all bounded analytic functions. The author is indebted to Professor Chevalley for the opportunity of reading a draft of the paper containing the proof.