SOME LIMIT THEOREMS

I. M. SHEFFER

1. **Introduction.** It is a classical result in the theory of trigonometric series that *if*

$$c_n \cos nx + d_n \sin nx \to 0 \qquad (n \to \infty) \tag{1.1}$$

for all (*real*) x *on a set of positive measure, then*

$$c_n \to 0, \qquad d_n \to 0. \tag{1.2}$$

Cantor proved this for the case that set $\{x\}$ is an interval, and Lebesgue established the result for a set of measure zero. A short proof is given by Hardy and Rogosinski.[1]

The following related result was proved and used by Szász.[2] *If*

$$a_n \sin nx + b_n \sin (n+1)x \to 0 \tag{1.3}$$

on a (*real*) *set* $\{x\}$ *of positive measure, then*

$$a_n \to 0, \qquad b_n \to 0. \tag{1.4}$$

Relations (1.1) and (1.3) can be put into complex form. For example, (1.1) becomes

$$a_n \exp \{nx\} + b_n \exp \{-nx\} \to 0, \tag{1.5}$$

with the conclusion that

$$a_n \to 0, \qquad b_n \to 0. \tag{1.6}$$

Here exp $\{u\}$ is defined by

$$\exp \{u\} \equiv e^{iu} \qquad (i = (-1)^{1/2}). \tag{1.7}$$

Our purpose in the present work is to extend the conclusions of the above-mentioned results to combinations more general than (1.3), (1.5). Thus in §2 we go from two terms to *k terms* and generalize the exponents; in §3 the coefficients of the exponentials are permitted

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[1] Hardy and Rogosinski, *Fourier series* (Cambridge Tracts in Mathematics and Mathematical Physics, no. 38), Theorem 92, p. 84.

[2] Otto Szász, *On Lebesgue summability and its generalization to integrals*, Amer. J. Math. vol. 67 (1945) pp. 389–396, especially Lemma 2, p. 395. Dr. Szász has informed me that, with the intention of using it in work on trigonometric series, he has proved (but not published) a generalization of (1.3), namely where the left side of (1.3) is replaced by the expression $\sum_{s=n}^{n+k} a_s e^{isx}$.