THE CRITICAL POINTS OF LINEAR COMBINATIONS OF HARMONIC FUNCTIONS

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In various extremal problems of function theory the critical points of linear combinations of Green's functions and harmonic measures are of significance.[1] The object of the present note is to indicate some of the more immediate results concerning the location of such critical points, for both simply and multiply connected regions.

The first configuration to be studied is a simple one. In the z-plane let C be the unit circle and let α_1 be an arc of C, whose initial and terminal points are a_1 and b_1 respectively, the positive direction chosen as counterclockwise. Let $\omega(z, \alpha_1, R)$ denote generically the harmonic measure of α_1 in the point z of R with respect to the region R; that is to say, ω is the function which is harmonic and bounded in R, continuous in the corresponding closed region except in the end points of α_1, equal to unity in the interior points of α_1 and to zero in the interior points of the boundary arcs complementary to α_1. The reader will verify the equation

$$\omega(z, \alpha_1, |z| < 1) = \frac{1}{\pi}\left[\arg(z - b_1) - \arg(z - a_1) - \frac{1}{2}\alpha_1\right]. \tag{1}$$

If ζ is a point interior to $C: |\zeta| < 1$, Green's function for $R: |z| < 1$ with pole in the point ζ can be written

$$g(z, \zeta, R) = \log |1 - \bar{\zeta} z| - \log |z - \zeta|. \tag{2}$$

An arbitrary linear combination of g and ω with real constant coefficients is $U(z) = \lambda g + \mu\omega$, which is the real part of the analytic function

$$f(z) = \lambda \log \frac{1 - \bar{\zeta} z}{z - \zeta} + \frac{\mu}{i\pi}\left[\log(z - b_1) - \log(z - a_1) - \frac{i\alpha_1}{2}\right].$$

The critical points of $U(z)$ are precisely the critical points of $f(z)$, namely the zeros of

$$if'(z) = \frac{i\lambda}{z - 1/\bar{\zeta}} - \frac{i\lambda}{z - \zeta} + \frac{\mu}{\pi}\left[\frac{1}{z - b_1} - \frac{1}{z - a_1}\right]. \tag{3}$$

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[1] See, for instance, M. Schiffer, Amer. J. Math. vol. 68 (1946) pp. 417–448; L. V. Ahlfors, Duke Math. J. vol. 14 (1947) pp. 1–11.