ON THE ROOTS OF A POLYNOMIAL AND ITS DERIVATIVE

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Let $r_1, r_2, \cdots, r_n$ be the roots of a polynomial $f(z)$ with complex coefficients, and let $R_1, R_2, \cdots, R_{n-1}$ be the roots of its derivative. N. G. de Bruijn[1] has proved that

$$(1) \qquad \frac{1}{n}\sum_{j=1}^{n} | I(r_j) | \geqq \frac{1}{n-1}\sum_{j=1}^{n-1} | I(R_j) |,$$

when $f(z)$ has real coefficients; he raises the question whether this holds in general. We prove that this inequality holds when $f(z)$ has complex coefficients; also that (1) is an equality only when the roots of $f(z)$ are not both above and below the real axis. An immediate consequence of this is that if $D_l(z)$ represents the (positive) distance from z to any straight line l in the complex plane, then

$$(2) \qquad \frac{1}{n}\sum_{j=1}^{n} D_l(r_j) \geqq \frac{1}{n-1}\sum_{j=1}^{n-1} D_l(R_j),$$

with the equality holding only when the r_j are not located on both sides of l. Further, if for any point A in the complex plane, $D_A(z)$ represents the distance from z to A, then

$$(3) \qquad \frac{1}{n}\sum_{j=1}^{n} D_A(r_j) \geqq \frac{1}{n-1}\sum_{j=1}^{n-1} D_A(R_j),$$

with the equality holding only when all the r_j lie on a half line emanating from A.

If A is taken as the origin, we have

$$\frac{1}{n}\sum_{j=1}^{n} | r_j |^m \geqq \frac{1}{n-1}\sum_{j=1}^{n-1} | R_j |^m$$

with $m=1$. This inequality, with $m=1, 2, 3, \cdots$, has been established by H. E. Bray[2] for the special case in which $f(z)$ is a real

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[1] *On the zeros of a polynomial and of its derivative*, K. Akademie van Wetenschappen, Proceedings vol. 49 (1946) pp. 1037–1044. *Added in proof*: In a second paper by de Bruijn and T. A. Springer, *On the zeros of a polynomial and of its derivative* II, Ibid. vol. 50 (1947) pp. 264–270, the results of the present paper are obtained, and the inequality following (3) is obtained for any $m \geqq 1$.

[2] *On the zeros of a polynomial and its derivative*, Amer. J. Math. vol. 53 (1931) pp. 864–872.