AN INEQUALITY CONCERNING POLYHEDRA[1]

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In 1897, at the mathematical competition of the Loránd Eötvös Mathematical and Physical Society, Professor L. Fejér, at the time still a student, noted the following interesting corollary of a well known elementary geometrical theorem of Euler:[2]

If R denote the radius of the circumscribed circle and r the radius of the inscribed circle of a given triangle, then

$$R \geqq 2r. \tag{1}$$

This is easily established, since according to the theorem of Euler mentioned above, if d denotes the distance between the centers of the circumscribed and inscribed circles, then

$$d^2 = R^2 - 2rR.$$

It follows that $R^2 - 2rR \geqq 0$, and therefore $R \geqq 2r$. Equality holds only if the two circles are concentric, that is, if the triangle is equilateral.

The problem of generalizing the above result to space was proposed by Professor L. Fejér. A young mathematician, I. Ádám, deported to Germany during the war—where all traces of him have been lost—found and communicated to Professor Fejér in 1943 a very simple proof of the above extremum property of the equilateral triangle. His proof, which may be immediately generalized to space, runs as follows:

If ρ is the radius of the circle passing through the midpoints of the sides of the triangle, then $\rho = R/2$, and all that need be shown is that ρ is at least equal to the radius of the inscribed circle. This follows from the fact that the inscribed circle is the smallest among all circles which have common points with all three sides of the triangle. Such a circle is, namely, the circle inscribed in a homothetic triangle containing the original one.

Received by the editors June 15, 1947.

[1] Lecture held in the seminar of Professor L. Fejér in April 1946.

[2] T. Radó, *On mathematical life in Hungary*, Amer. Math. Monthly vol. 39 (1932) pp. 85–90; J. Kürschák, *Matematikai versenytételek*, Szeged, 1929. A sharper inequality than (1) is contained in a problem proposed by M. Schreier, Jber. Deutschen Math. Verein. vol. 45 (1935) p. 196. L. J. Mordell [Középiskolai Matematikai és Fizikai Lapok vol. 11 (1935) pp. 145–146, see also Amer. Math. Monthly vol. 44 (1937) p. 252] has proved an analogous better inequality conjectured by P. Erdös: If R_1, R_2, R_3 are the distances of an inner point in a triangle from the vertices and r_1, r_2, r_3 the distances from the sides then $R_1+R_2+R_3 \geqq 2(r_1+r_2+r_3)$.