STEINER'S FORMULAE ON A GENERAL S^{n+1}

CARL B. ALLENDOERFER

1. Introduction. Steiner's famous formulae on parallel curves and surfaces have attracted considerable interest recently, several mathematicians having developed various extensions of these theorems [3, 4, 6].[1] As stated by Steiner [5] these formulae have the following form:

THEOREM 1. *Let C be a convex curve in the plane of length L and area F, and let C_ρ be the curve parallel to C at a distance ρ from it (measured outward) with length L_ρ and area F_ρ; then*

$$L_\rho = L + 2\pi\rho, \qquad F_\rho = F + \rho L + \pi\rho^2.$$

THEOREM 2. *Let Σ be a convex surface in ordinary space of surface area S, enclosed volume V, and integrated mean curvature M; and let Σ_ρ be the surface parallel to Σ at a distance ρ from it (measured outward) with surface S_ρ and volume V_ρ; then:*

$$S_\rho = S + 2M\rho + 4\pi\rho^2, \qquad V_\rho = V + S\rho + M\rho^2 + 4\pi\rho^3/3.$$

We shall prove the following generalization of these results:

THEOREM 3. *Let S^{n+1} be a Riemann space of constant curvature, K, differentiable of class C^3 and complete in the sense of Hopf and Rinow. Let V^n be a hypersurface of S^{n+1} which is closed and bounding in S^{n+1} and of class C^3, and whose principal curvatures with respect to an outward normal are all negative. The area of V^n will be called A and its volume Vol. Its various mean curvatures (to be defined in §3) will be called M_i $(i=0, \cdots, n)$. Let V^n_ρ be a surface parallel to V^n at a distance measured along outward drawn geodesics where:*

$$\textit{for} \quad K > 0: 0 \leqq \rho \leqq \pi/2K^{1/2}; \quad \textit{and for} \quad K < 0: \rho \geqq 0.$$

Further let the area and volume of V^n_ρ be respectively A_ρ and Vol_ρ. *Then for $K>0$:*

$$A_\rho = \sum_{i=0}^{n} M_i (K^{-1/2} \sin [\rho K^{1/2}])^{n-i} (\cos [\rho K^{1/2}])^i,$$

$$\mathrm{Vol}_\rho = \mathrm{Vol} + \sum_{i=0}^{n} M_i \int_0^\rho (K^{-1/2} \sin [x^0 K^{1/2}])^{n-i} (\cos [x^0 K^{1/2}])^i dx^0;$$

Received by the editors April 18, 1947.

[1] Numbers in brackets refer to the references cited at the end of the paper.