ON THE SUM OF THE RELATIVE EXTREMA OF $|f(z)|$ ON THE UNIT CIRCLE

ROBERT BREUSCH

Summary and introduction. Dr. Erdös has made the following conjecture: "Let $f(z)=\prod_{\nu=1}^{n}(z-z_\nu)$, with $z_\nu=r_\nu e^{i\theta_\nu}$, $r_\nu\leqq 1$, $f(z)\neq z^n$. Let $z_\mu=e^{i\theta_\mu}$ ($\mu=1, 2, \cdots k$) be the points on the circumference of the unit circle, for which $|f(z)|$ is a relative extremum. Then $\sum_{\mu=1}^{k}|f(z_\mu)|\leqq 2^n$, and the equal sign applies only if $f(z)=(z-e^{i\theta_0})^n$." It will be proved that this is correct for large values of n, but incorrect for small ones.

With $z=e^{i\theta}$,

$$|f(z)|^2=\prod_{\nu=1}^{n}|e^{i\theta}-r_\nu e^{i\theta_\nu}|^2=\prod_{\nu=1}^{n}[1+r_\nu^2-2r_\nu\cos(\theta-\theta_\nu)],$$

or, with $t=\tan(\theta/2)$,

$$\begin{aligned}|f(z)|^2\equiv F(t)&=(1+t^2)^{-n}\\&\cdot\prod_{\nu=1}^{n}[(1+r_\nu^2+2r_\nu\cos\theta_\nu)t^2-4r_\nu\sin\theta_\nu t+(1+r_\nu^2-2r_\nu\cos\theta_\nu)],\end{aligned}$$

$$F(t)=\frac{At^{2n}+\cdots}{(1+t^2)^n},$$

$$F'(t)=\frac{(1+t^2)(2nAt^{2n-1}+\cdots)-2nt(At^{2n}+\cdots)}{(1+t^2)^{n+1}}=\frac{Bt^{2n}+\cdots}{(1+t^2)^{n+1}}.$$

It may be assumed for the moment that $|f(e^{i\theta})|$ is not an extremum for $\theta=\pi$, that is, $t=\infty$; otherwise the coordinate system may be rotated. Thus it is seen that the number of relative extrema of $|f(z)|$ on the unit circle cannot exceed $2n$.

If $|f(e^{i\theta})|$ is less than $2^n/2n$ for every real value of θ, then $\sum_{\mu=1}^{k}|f(e^{i\theta_\mu})|$ is obviously less than 2^n.

Assume now that $|f(z)|$ has a relative extremum at $z=1$, and that $|f(1)|\geqq 2^n/2n$. The proof then proceeds in the following steps: It will be shown, that for large values of n:

(1) All but $o(n)$ roots must be in a region R_1 of the unit circle close to $z=-1$.

(2) There is an arc T of the circumference of the unit circle, close to $z=+1$, such that the sum of all the extreme values of $|f(e^{i\theta})|$ with $e^{i\theta}$ not on T is $O(2^n/n)$.

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