CONGRUENCE PROPERTIES OF RAMANUJAN'S FUNCTION $\tau(n)$

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Introduction. With Ramanujan we define $\tau(n)$ by

$$\sum_{1}^{\infty} \tau(n) x^n = x \prod_{1}^{\infty} (1 - x^n)^{24} \qquad (|x| < 1).$$

Write $\sigma_k(n)$ for the sum of the kth powers of the divisors of n; $\sigma(n) = \sigma_1(n)$. It is known that[1]

$$\tau(n) \equiv n\sigma(n) \pmod 5,$$
$$\tau(n) \equiv \sigma(n) \pmod 3 \qquad \text{if } (n, 3) = 1.$$

The object of this note is to give proofs of the much stronger results:

(A) $$\tau(n) \equiv 5n^2\sigma_7(n) - 4n\sigma_9(n) \pmod{5^3}$$

when n is prime to 5;

(B) $$\tau(n) \equiv (n^2 + k)\sigma_7(n) \pmod{3^4}$$

when n is prime to 3 and where $k=0$ if $n \equiv 1(3)$, $k=9$ if $n \equiv 2(3)$.

1. Some lemmas.

LEMMA 1. *We have*

$$\sum u\sigma_3(u)\sigma_5(v) \equiv \sum \sigma(u)\sigma(v) - P(n) \pmod 5$$

where

$$P(n) = \sum_{u \equiv 0 \pmod 5} \sigma(u)\sigma(v)$$

where $u+v=n$; $u, v \geqq 1$ *in all three sums* $(\sum)$.

PROOF. We have

(1) $$u\sigma_3(u)\sigma_5(v) \equiv 0 \pmod 5 \quad \text{when} \quad u \equiv 0(5);$$

when $(u, 5) = 1$ we have

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[1] The first of these is proved in Hardy's *Ramanujan* (Cambridge, 1940); the second by Gupta in J. Indian Math. Soc. vol. 9 (1945) pp. 59–60. In what follows we refer to Ramanujan's *Collected papers* (Cambridge, 1927) by the letters RCP. We have also proved that $\tau(n) \equiv \sigma_{11}(n) \pmod{2^8}$ if n is odd; this result has been accepted for publication in J. London Math. Soc.