HARMONIC TRANSFORMATION THEORY OF ISOTHERMAL FAMILIES

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1. Harmonic transformations. Let a transformation T

$$X + \phi(x, y), \qquad Y = \psi(x, y), \tag{1}$$

with the jacobian

$$J = \phi_x\psi_y - \phi_y\psi_x \neq 0, \tag{2}$$

be such that the components ϕ and ψ satisfy the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0, \qquad \psi_{xx} + \psi_{yy} = 0, \tag{3}$$

in a certain region of the real (or complex) cartesian plane. We shall term any such correspondence T a *harmonic transformation.*

The harmonic transformations form an infinite set (H) of $\infty^{4f(1)}$ correspondences since they are defined essentially by four independent functions of a single variable. The totality of harmonic correspondences of course do not constitute a group.[1]

A subset of the class (H) of harmonic transformations is the conformal group. The components ϕ and ψ of a conformal map are of course conjugate-harmonic, that is, they satisfy the direct or reverse Cauchy-Riemann equations

$$\phi_x = \pm \psi_y, \qquad \phi_y = \mp \psi_x. \tag{4}$$

Thus a harmonic transformation is conformal if and only if its components are conjugate-harmonic. In general, the components of a harmonic transformation are not interrelated in any way whatsoever.

We have proved that in the real domain the only groups contained in the infinite set (H) of $\infty^{4f(1)}$ harmonic transformations are the group of $\infty^{2f(1)}$ conformal maps, the group of ∞^6 affinities, and the subgroups of these two. In the imaginary domain, we have found in addition two extra infinite groups each consisting of $\infty^{1+2f(1)}$ transformations.

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[1] Harmonic functions and harmonic transformations appear in the theory of minimal surfaces and the Plateau problem. In particular see the fundamental papers of Schwarz and Douglas. Schwarz discusses the case where the jacobian of a harmonic transformation vanishes and studies possible singularities. See *Abhandlungen*, vol. 1, p. 293. Douglas studies the inverse of harmonic transformations. See abstracts in Bull. Amer. Math. Soc. vol. 49 (1943) and vol. 50 (1944).