RATIONAL HARMONIC CURVES

EDWARD KASNER AND JOHN DECICCO

1. **Introduction.** We shall study curves related to rational fractional functions of a complex variable. This will generalize known results of curves related to rational integral functions.

Curves defined by setting the real part of a polynomial (rational integral function) in the complex variable $u=x+iy$ equal to zero are well known. These had been studied initially by Briot and Bouquet, and Bôcher; and finally characteristic properties have been given by Kasner. These curves have been called *algebraic potential curves* by Kasner, and this term is employed in later papers by Loria and in the German *Encyclopedia*. But we shall find it more convenient to use the term: *polynomial harmonic or polynomial potential curves.*

We define a *rational harmonic or rational potential curve* to be the locus obtained by setting the real part of a rational fractional function of a complex variable $u=x+iy$ equal to zero. The class of rational harmonic curves of course includes the class of polynomial harmonic curves.

We shall obtain various geometric properties of rational harmonic curves. These generalize corresponding results of Briot and Bouquet, and Kasner concerning the polynomial potential curves. We shall prove that *the real asymptotes of a rational potential curve are concurrent and make equal angles with one another*; *the remaining asymptotes are minimal.* This condition is only necessary but not sufficient. We do find a characteristic property of rational potential curves by studying the related focal properties. In the final part of our paper, we study the Schwarzian reflection with respect to a rational harmonic curve. *The satellite of a rational harmonic curve is itself.* This result gives the largest known class of self-satellite algebraic curves.

2. **Theorems of Briot and Bouquet, and Kasner concerning polynomial potential curves.** For purposes of contrast, certain theorems concerning polynomial harmonic curves will be stated.

The theorem of Briot and Bouquet concerning the asymptotes of a polynomial harmonic curve is as follows.[1]

The n asymptotes of a polynomial potential curve of degree n are all

Presented to the Society, December 29, 1946; received by the editors October 10, 1946.

[1] Briot and Bouquet, *Theorie des fonctions elliptiques*, vol. 4, Paris, Gauthier-Villar, 1875, chap. 2, p. 226. See also Bôcher, Göttingen prize memoir.