A TERNARY OPERATION IN DISTRIBUTIVE LATTICES

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It can be easily seen that the *graph* [1, p. 9],[1] of the Boolean algebra B^n of 2^n elements (consisting of the vertices and edges of an n-cube) has $2^n(n!)$ "link-automorphisms," whereas B^n has only $(n!)$ lattice-automorphisms. In an unpublished book,[2] one of us has developed new operations in B^n and other distributive lattices, which admit such a wider group of invariance. The purpose of this note is to show the role of the symmetric and self-dual ternary operation [1, p. 74]

$$(1)\qquad \begin{aligned}(a, b, c) &= (a \cap b) \cup (b \cap c) \cup (c \cap a)\\ &= (a \cup b) \cap (b \cup c) \cap (c \cup a)\end{aligned}$$

in a general distributive lattice L, with reference to the wider group of symmetries which it admits.

THEOREM 1. *In any metric distributive lattice* [1, p. 41], *the following conditions are equivalent*: (i) $a\cap b \leqq x \leqq a\cup b$, (ii) $|a-x|+|x-b| = |a-b|$, (iii) $(a, x, b)=x$.

PROOF. V. Glivenko [3, p. 819, Theorem V] has shown the equivalence of (i) and (ii); condition (i) says that x is metrically "between" a and b in the sense of Menger. But now if $a\cap b \leqq x \leqq a\cup b$, then $(a, x, b)=(a\cap b)\cup(b\cap x)\cup(x\cap a)=(a\cap b)\cup[x\cap(a\cup b)]=x$. Conversely, if $(a, x, b)=x$, then

$$x = (a \cap b) \cup (b \cap x) \cup (x \cap a) \geqq a \cap b,$$

and dually, $x\leqq a\cup b$. Hence (i) and (iii) are equivalent.

DEFINITION. The segment joining a and b is the set of x satisfying any (hence all) of the conditions of Theorem 1 (cf. Duthie [4]); we denote it $[a, b]$.

THEOREM 2. *The element* (a, b, c) *is the intersection of the sets* $[a, b]$ $[b, c]$ $[c, a]$.

PROOF. This is obvious from condition (i) and formula (1).

COROLLARY 1. *The element* (a, b, c) *minimizes*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] S. A. Kiss, *Transformations on lattices and structures of logic*, Bull. Amer. Math. Soc. Abstract 52-1-4.