NOTE ON HADAMARD'S DETERMINANT THEOREM

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Introduction. We shall call a square matrix A of order n an Hadamard matrix or for brevity an H-matrix, if each element of A has the value ± 1 and if the determinant of A has the maximum possible value $n^{n/2}$. It is known that such a matrix A is an H-matrix [1][1] if, and only if, $AA' = nE_n$ where A' is the transpose of A and E_n is the unit matrix of order n. It is also known that, if an H-matrix of order $n > 1$ exists, n must have the value 2 or be divisible by 4. The existence of an H-matrix of order n has been proved [2, 3] only for the following values of $n > 1$: (a) $n = 2$, (b) $n = p^h + 1 \equiv 0 \bmod 4$, p a prime, (c) $n = m(p^h+1)$ where $m \geqq 2$ is the order of an H-matrix and p is a prime, (d) $n = q(q-1)$ where q is a product of factors of types (a) and (b), (e) $n = 172$ and for n a product of any number of factors of types (a), (b), (c), (d) and (e).

In this note we shall show that an H-matrix of order n also exists when (f) $n = q(q+3)$ where q and $q+4$ are both products of factors of types (a) and (b), (g) $n = n_1 n_2 (p^h+1)p^h$, where $n_1 > 1$ and $n_2 > 1$ are orders of H-matrices and p is an odd prime, and (h) $n = n_1 n_2 m(m+3)$ where $n_1 > 1$ and $n_2 > 1$ are orders of H-matrices and m and $m+4$ are both of the form p^h+1, p an odd prime.

It is interesting to note the presence of the factors n_1 and n_2 in the types (g) and (h) and their absence in the types (d) and (f). Thus, if p is a prime and $p^h + 1 \equiv 0 \bmod 4$, an H-matrix of order $p^h(p^h+1)$ exists but, if $p^h + 1 \equiv 2 \bmod 4$, we can only be sure of the existence of an H-matrix of order $n_1 n_2 p^h(p^h+1)$ where $n_1 > 1$ and $n_2 > 1$ are orders of H-matrices. This is analogous to the simpler result that, if $p^h + 1 \equiv 0 \bmod 4$ an H-matrix of order p^h+1 exists but, if $p^h+1 \equiv 2 \bmod 4$, we can only be sure of the existence of an H-matrix of order $n(p^h+1)$ where $n > 1$ is the order of an H-matrix.

We shall denote the direct product of two matrices A and B by $A \cdot B$ and the unit matrix of order n by E_n.

Theorems on the existence of H-matrices. If a symmetric H-matrix of order $m > 1$ exists, there exists an H-matrix H of order m with the form

$$H = \begin{pmatrix} 1 & e \\ e' & D \end{pmatrix},$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.