SOME ASYMPTOTIC FORMULAS FOR MULTIPLICATIVE FUNCTIONS

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The present paper contains several asymptotic formulas for the sum of multiplicative functions. A function $f(n)$ is called multiplicative if $f(a\cdot b)=f(a)\cdot f(b)$ for $(a, b)=1$. We assume $f(n)>0$. In this paper $f(n)$, $f_1(n)$ will always denote multiplicative functions. First we prove the following theorem.

THEOREM 1.[1] *Assume that the two series*

$$(1)\qquad -\sum_{p,\alpha}\frac{f(p^\alpha)-1}{p^\alpha},\qquad \sum_{p,\alpha}\frac{(f(p^\alpha)-1)^2}{p^\alpha},$$

converge; then $f(n)$ has a mean value, that is,

$$\lim_{n\to\infty}\frac{1}{n}\sum f(m)$$

exists and is not equal to zero.

This result was conjectured in a slightly more special form at the end of my paper *Some remarks on additive and multiplicative functions*.[2]

REMARK. The convergence of (1) is the necessary and sufficient condition for the existence of the distribution function of $f(n)$.[3]

For the sake of simplicity we assume $f(p^\alpha)=f(p)$. Then we prove

$$(2)\qquad \lim_{n\to\infty}\frac{1}{n}\sum_{m\leqq n}f(m)=\prod_p\left(1+\frac{f(p)-1}{p}\right).$$

It easily follows from (1) that the product on the right side of (2) converges and thus the value of the limit is not 0.

We easily obtain from (1) that for every $\epsilon>0$

$$\sum_{|f(p)-1|>\epsilon}\frac{1}{p}<\infty.$$

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[1] This result generalizes a result of Wintner, Amer. J. Math. vol. 67 (1945) pp. 481–485.

[2] Bull. Amer. Math. Soc. vol. 52 (1946) pp. 527–537.

[3] P. Erdös and A. Wintner, Amer. J. Math. vol. 61 (1939) pp. 713–721. See also footnote 2.