A NOTE ON THE MINIMUM MODULUS OF A CLASS OF INTEGRAL FUNCTIONS

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A well known theorem due to Littlewood, Wiman, and Valiron[1] states that for any integral function of order less than one-half,

$$\log m(r) > (\text{a positive constant}) \log M(r),$$

on a sequence of circles of indefinitely increasing radius. I consider in this note a class of integral functions which have this property and prove the following theorem.

THEOREM 1. *Hypothesis*:

(1) (R_n) *is any sequence of positive numbers such that* $R_1 > 1$, $R_n/R_{n-1} \geqq \lambda > 1$.

(2) (p_n) *is any sequence of positive integers.*

(3) $a_{11}, a_{12}, \cdots, a_{1p_1}, a_{21}, \cdots, a_{2p_2}, \cdots$ *are a set of points such that* $0 < |a_{11}| \leqq |a_{12}| \leqq \cdots$ *and such that a finite number* $a_{n1}, \cdots, a_n p_n$ *lie inside the ring* $(R_n - R_n^\alpha < |z| < R_n)$ *where* $0 < \alpha < 1$.

(4) μ_n *is a sequence of positive integers such that* $\sum_1^\infty p_n/\beta^{\mu_n}$ *is convergent,* β *being any constant greater than one.*

(5) *The exponent of convergence of the points*

$$a_{nr} \exp (2\pi i v/\mu_n),$$

where $r = 1, 2, \cdots, p_n$; $v = 0, 1, 2, \cdots, \mu_n - 1$; $n = 1, 2, 3, \cdots$, is ρ $(0 \leqq \rho < \infty)$.

(6)[2] *Lower bound* $\{\mu_n\} \geqq 1 + \rho$.

Conclusion:

(7) *The canonical product*

$$f(z) = \prod_{n=1}^{\infty} \prod_{s=1}^{p_n} \left\{1 - \frac{z^{\mu_n}}{a_{ns}^{\mu_n}}\right\} \tag{8}$$

formed with these points as zeros is of order ρ; *and the values of* $r = |z|$ *for which the inequality*

$$m(r, f) > CM(r, f),$$

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[1] G. Valiron, *Lectures on the general theory of integral functions*, pp. 128–130.

[2] It is possible to choose R_n, p_n, and so on, satisfying the conditions (1) to (6). Example: $R_n = 2^{2n}$; $p_n = n^2 2^n$; $\mu_n = 2^n$. Here $\rho = 1$.