NOTE ON THE DERIVATIVES OF FUNCTIONS ANALYTIC IN THE UNIT CIRCLE

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In a recent paper,[1] W. Seidel and the present writer established distortion theorems for various classes of functions analytic in the unit circle; more explicitly, established relations between the derivatives of functions and their radii of univalence and of p-valence, with particular reference to behavior as a point approaches the circumference. Classes studied in detail were functions respectively univalent, bounded, omitting two values, p-valent, and having a bounded radius of univalence. The last-named class clearly includes the class of functions $f(z)$ each analytic in $|z| < 1$ and transforming $|z| < 1$ onto a Riemann configuration of finite area. It is the primary object of the present note to study this included class to the same end more effectively by other methods. Terminology and notation are uniform with the paper referred to; unless otherwise specified, references in the present note are to that paper; we shall also have occasion to use the results of a subsequent paper by Loomis.[2]

We denote by K_M the class of functions $f(z)$ analytic in $|z| < 1$ and transforming the region $|z| < 1$ into a region whose area (counted according to the multiplicity of covering) is not greater than πM^2:[3]

$$\iint_{|z|<1} |f'(z)|^2 dS \leqq \pi M^2. \tag{1}$$

If we set $f(z) \equiv \sum_0^\infty a_n z^n$, inequality (1) becomes

$$\sum_1^\infty n |a_n|^2 \leqq M^2, \tag{2}$$

whence we have

$$|a_n| \leqq \frac{M}{n^{1/2}}. \tag{3}$$

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[1] Trans. Amer. Math. Soc. vol. 52 (1942) pp. 128–216.

[2] Bull. Amer. Math. Soc. vol. 48 (1942) pp. 908–911.

[3] *Added in proof.* This class of functions has recently been studied by P. Montel (Publicaciones del Instituto de Matemática de la Universidad Nacional del Litoral, Rosario, vol. 6 (1946) pp. 273–286), who obtains sharp inequalities improving (12) and (13) below, but does not prove any of the italicized theorems of the present note. Compare also T. H. Gronwall, Ann. of Math. vol. 16 (1914) pp. 72–76.