A CHARACTERIZATION OF MINKOWSKIAN GEOMETRY

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Menger called a finitely-compact metric space in which any two distinct points are contained in exactly one set congruent to a euclidean straight line a straight-line space (S.L. space) [1].[1] He observed that in such spaces a segment (or vector) addition can be defined in a very simple way (see §2 of the present paper) which satisfies all requirements for an abelian group except, in general, the associative law. Menger then put the question of determining those geometries in which the associative law holds. It is the purpose of the present note to show that the Minkowskian geometries furnish the answer.[2]

1. **Straight-line spaces.**[3] By a straight-line (S.L.) in a metric space R is understood a continuous curve which may be mapped congruently onto the real axis, that is, which admits of a parametrization $P(\tau)$, $-\infty<\tau<\infty$, with $P(\tau_1)P(\tau_2)=|\tau_1-\tau_2|$. The numbers τ are the *isometric coordinates* of the points of the S.L. A *segment* joining two points $P(\tau_1)$, $Q(\tau_2)\subset R$, written (PQ), is a congruent image of the closed interval between τ_1 and τ_2.

An *S.L. space* is one in which any pair of its points X, Y is contained in a unique S.L. $[XY]$. The following conditions are sufficient for an S.L. space: the space is (1) metric, (2) finitely-compact, (3) convex, (4) externally convex, (5) if (XYZ_1), (XYZ_2), and $YZ_1=YZ_2$, then $Z_1=Z_2$. We shall suppose, hereafter, that these conditions hold.

If $X_n\to X$, $Y_n\to Y$ and $X\neq Y$ then $[X_nY_n]\to[XY]$.

According to Menger M is an *internal center* of X and Y, written $M(X, Y)$, if (XMY) and $XM=MY$. Also, if (EXY) and $EX=XY$, E is an *external center* of X and Y, written $E(X, Y)$ or $(Y, X)E$. In an S.L. space every point-pair has a unique internal center and two external centers.

2. **Addition of segments in S.L. spaces.** Let (OX), (OY) be two arbitrary segments in R having the end point O in common. We shall call the segment (OP) a *sum* of the two given segments, written

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] The above problem, together with a conjecture as to the answer, was suggested by H. Busemann.

[3] For explanations of the notations used here, as well as proofs of the statements of this section, see Busemann [2, chaps. I and III].