ABSOLUTE AND UNCONDITIONAL CONVERGENCE

M. S. MACPHAIL

A series $\sum_{n=1}^{\infty} x_n$ whose terms are elements of a Banach space B is said to converge *absolutely* if $\sum |x_n|$ converges, *unconditionally* if every rearrangement converges. If a series converges absolutely it converges unconditionally; absolute convergence and unconditional convergence are equivalent in any finite-dimensional Euclidean space, whereas in L^2, for instance, there are simple examples of series which converge unconditionally but not absolutely. The question in what spaces the two definitions are equivalent is left open by Banach [**1**, p. 240].[1] It may be conjectured that unconditional convergence does not imply absolute convergence in any infinite-dimensional space (cf. [**2**, p. 30]), but the problem does not seem to have been explicitly treated. The purpose of this note is to obtain a criterion for the equivalence of the two notions in a given Banach space. As examples of its use we shall show that unconditional convergence does not imply absolute convergence in the spaces L and l. The result for L is already known (see for example [**3**, p. 45]), but the result for l has not, to the author's knowledge, been stated elsewhere.

We shall need the following definitions. Let S be any (finite) sequence of elements $\xi_1, \xi_2, \cdots, \xi_p$ in B. We do not require that the ξ_i be all distinct, and we shall understand by *addition*, S_1+S_2, the mere adjoining of the two sequences. Thus we might write $(1, 2, 2, 3)+(3, 2, 1)=(1, 2, 2, 3, 3, 2, 1)$. By cS (c real) we understand the sequence $c\xi_1, c\xi_2, \cdots, c\xi_p$. We shall use two norms,

$$|S| = \sum |\xi_i|, \qquad |S|^* = \sup_{\sigma} \left| \sum_{\sigma} \xi_i \right|,$$

where σ is any subset of $1, 2, \cdots, p$. It is easily verified that $|S_1+S_2| = |S_1|+|S_2|$, $|S_1+S_2|^* \leqq |S_1|^*+|S_2|^*$, $|cS| = |c|\,|S|$, and $|cS|^* = |c|\,|S|^*$. Define further, when $|S| \neq 0$,

$$G(S) = |S|^*/|S|.$$

Note that $0 < G(S) \leqq 1$. Let $g = \inf G(S)$, taken over all sequences $S \subset B$.

THEOREM.[2] *If $g=0$, unconditional convergence does not imply absolute convergence in B. If $g>0$, the two are equivalent.*

Received by the editors July 25, 1946.

[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] Suggested to the author by Garrett Birkhoff, in conversation.