IRREDUCIBLE REPRESENTATIONS OF OPERATOR ALGEBRAS

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1. **Introduction.** In this paper we investigate the representation on (not necessarily separable) Hilbert spaces of a suitably closed algebra of operators on a Hilbert space, with regard to the existence of irreducible representations and the connection between representations and states of the algebra (we use the term "state" to mean a complex-valued linear function on the algebra which is real and non-negative on positive semi-definite operators, and which satisfies a certain normalizing condition—more commonly, this is called an expectation value in a state). Our results are, first, that there is a mutual correspondence between "normal" representations (those for which there is an element of the Hilbert space whose transforms under the representation operators span the space) and states of the algebra, in which irreducible representations correspond to pure states (a state is pure if it is not a linear combination with positive coefficients of two other states), and second, that there exists a complete set of irreducible representations for a given algebra (or alternatively, a complete set of pure states).

The operator algebras which we consider are assumed to be: (1) closed in the uniform topology (in which the distance between two operators is defined as the bound of their difference), (2) self-adjoint (that is, if an operator is in the algebra then so is its adjoint); we call such an algebra a C^*-algebra. By a representation of a C^*-algebra $\mathcal{A}$ we mean a function ϕ on $\mathcal{A}$ to the collection of bounded operators on a Hilbert space which is an algebraic homomorphism, continuous, and such that for $U \in \mathcal{A}$, $\phi(U^*) = (\phi(U))^*$, the asterisks denoting adjoints. The representation is called irreducible if the Hilbert space on which $\mathcal{A}$ is represented contains no nontrivial closed invariant subspace. A collection of representations of $\mathcal{A}$ is called complete if the only element of $\mathcal{A}$ which is mapped into zero by every representation is zero.

Our results make possible a broader and more rigorous treatment of certain parts of quantum mechanics, particularly of the principle that the spectral values of an observable are determined by the behavior of the observable in the irreducible representations of the operator algebra describing the physical system in question. The original mathematical model for the observables in quantum theory was the

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