# TOPOLOGICAL METHODS IN THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

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1. **Introduction.** Some indication of the reasons for which the authors have undertaken the study of topological methods in the theory of functions of a complex variable is appropriate.

The modern theory of meromorphic functions has distinguished itself by the fruitful use of the instruments of modern analysis and in particular by its use of the theories of integration. Its success along the latter line has perhaps diverted attention from some of the more finitary aspects of the theory which may be regarded as fundamental. In particular, the classical use of the Cauchy integral

$$\frac{1}{2\pi i}\int_C \frac{f''(z)}{f'(z)}\,dz$$

to find the difference between the number of zeros and poles of $f'(z)$ within $C$ is in a sense statistical and ignores important extremal properties of the boundary values such, for example, as the extremal values of $|f(z)|$. In addition, its application requires the existence of $f''(z)$, at least almost everywhere on $C$, and the nonvanishing of $f'$ on $C$. As we shall see there is a topological substitute for this integral under much weaker conditions on $f$. The needs of the classical theory usually require that the curve $C$ be regular or rectifiable. The topological analogue makes use of Jordan curves whose images under $f$ are "locally simple." See §3.

The study of analytic functions is of course paralleled by the study of harmonic functions. An example of a type of mathematical phenomena which the classical theory has failed to reach is given by a harmonic function $U(x, y)$ with infinitely many critical points on a Jordan region $R$. If $U(x, y)$ is continuous on $\overline{R}$, the closure of $R$, there are topological or group theoretic relations between the critical points of $U$ on $R$ and the relative minima of $U$ on the boundary of $R$. These relations take the form of isomorphisms, termed "causal" by the authors, between an appropriately defined group of relative 1-cycles with the $U$-"heights"[1] of the respective "saddle points" of $U$, and a

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[1] The $U$-height of a cycle is the maximum value of $U(x, y)$ on the cycle.